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PRACTICAL TREATISE

ON

ALGEBRA,

DESIGNED FOR THE USE OF STUDENTS

IN

HIGH SCHOOLS AND ACADEMIES.

BY BENJAMIN GREENLEAF, A.M.,
AUTHOR OF THE "NATIONAL ARITHMETIC," ETC.

Ninth Improved Stereotype Edition.

BOSTON:
PUBLISHED BY ROBERT S. DAVIS & CO.
NEW YORK: GEO. F. COOLEDGE & BROTHER, AND D. BURGESS & CO.
PHILADELPHIA: LIPPINCOTT, GRAMBO & CO.
ST. LOUIS: JOHN HALSALL.
1855.

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PREFACE.

THE following Treatise is designed to present a system of theoretical and practical Algebra. It is intended to be both elementary and comprehensive, and adapted to the wants of beginners, as well as those who are advanced in the study.

In the course of his labors the author has consulted the most approved European treatises on the subject, and availed himself of whatever he thought might add to the interest and usefulness of his work.

It has been the aim of the author, throughout his investigations, to give to it a practical character, that those who study it may know how to apply their knowledge to useful purposes.

The demonstrations connected with the several Roots, will greatly aid those who wish for a complete and thorough knowledge of Evolution in Arithmetic.

The method of solving Cubic Equations by completing the square, the author believes, will be very useful. This method will not apply to all problems; but, wherever it will apply, it not only very much abridges the labor, but the result is perfect accuracy, which is not always the fact by the common method of approximation. The Table of Logarithms at the end of the work, will be often found convenient and useful.

The examples, of which a large number have been placed under each Rule, are intended to be neither too numerous nor too difficult; and all who may use the work, either by themselves or in connection with a class, are advised to solve all the problems, in the order in which they are given. No labor on the part of the pupil will be productive of more intellectual and practical benefit. The answers to several questions have been designedly omitted, that the pupil may try his skill as upon an original problem.

One who has a thorough knowledge of Arithmetic, will find the study of Algebra a most pleasing, and, generally, not a difficult task. As a mental exercise, it is admirable for its effect upon the logical powers of the mind, assisting one to think and reason closely and conclusively. As Mr. Locke has remarked, in his Essay on the Human Understanding, "Nothing teaches a man to reason so well as Mathematics, which should be taught to all those who have time and opportunity, not so much to make them mathematicians, as to make them reasonable creatures."

BENJAMIN GREENLEAF.

BRADFORD, *January* 28, 1852.

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IN revising this work for a second edition, the author has made such changes and additions as he believed would better adapt it to its purpose. Every part of it has been carefully and critically examined, and many portions have been entirely re-written. In a few cases, where improvement in that respect seemed desirable, the arrangement of articles has been somewhat altered.

The new articles which have been inserted, will, it is hoped, add materially to the interest, as well as to the value, of the treatise. The theory of Equations has been more fully developed, and illustrated by a variety of carefully prepared examples. A brief space has been given to Indeterminate Analysis, a subject which, though usually omitted in elementary works on Algebra, the author believes to be one of no small practical importance. It gives the student the command of a class of problems which cannot possibly be solved by the rules of Arithmetic, nor by the more familiar principles of Algebra.

In the revision of the work, the author has availed himself of the suggestions of several teachers who have used it as a text-book since its first publication; and he would take this opportunity to express his gratitude for their kindness.

April 26, 1853.

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ALGEBRA.

SECTION I.

DEFINITIONS AND NOTATIONS.

ARTICLE **1.** ALGEBRA is the art of computing by symbols.

2. In Arithmetic we represent quantities and perform calculations by figures, each of which has a known and definite value.

3. In Algebra we employ the letters of the alphabet, and other characters, the value of which is either known or unknown, according to the conditions of the problems.

4. Those quantities whose values are given are called *known* quantities; and those whose values are not given are *unknown* quantities.

5. The symbols used to denote known quantities are, generally, the first letters of the alphabet in the small or Italic character, as a, b, c, d, &c.; and those used to denote unknown quantities, the last letters, as w, x, y, z.

6. In addition to the above, which are the more common symbols, capital letters may be used, as A, B, C, D, &c., or letters of the Greek alphabet, as α, β, γ, δ, ε, ζ, &c. In extensive operations, the use of these, or some other suitable characters, is sometimes very convenient.

7. Sometimes quantities are expressed in Algebra, as in Arithmetic, by figures instead of letters.

8. When a quantity is doubled or trebled, or multiplied any number of times, the number of multiplications is usually expressed by a numerical figure or figures. Thus, let a denote a certain quantity, and $2a$ will denote twice the same quantity, $3a$ three times the same quantity, &c.

9. The figures or letters prefixed to any symbol, and denoting the number of times the quantity represented by the symbol is taken, are called the *coefficient*. Thus, in $4a$, $7b$, and $4ax$, the coefficients are 4, 7, and $4a$.

10. A quantity which has no figure prefixed to it is considered as having a unit for its coefficient. Thus, a is the same as $1a$.

11. Quantities represented by the same symbol or letters, and of the same power, are called *like* quantities; and those represented by different symbols or letters, or by the same letter of different powers, *unlike* quantities. Thus, $3a$, $4a$, and $5a$, are like quantities; and $3a$, $4b$, and $5c$, unlike quantities. In like manner, $3ab$, $4ab$, and $6ab$, are like quantities; and $3ab$, $4ac$, $5dc$, and $6mn$, are unlike quantities.

12. Besides the symbols and figures used to denote quantity, there are certain signs, which are used to express the different relations between quantities, and the operations to which these quantities are subjected. These signs are the same as are often employed in Arithmetic, but, in Algebra, they are indispensable.

13. The sign $=$ is that of *equality*, and denotes that the two quantities between which it is placed are equal to each other. Thus, $a=2b$ signifies that a is equal to $2b$.

14. The sign $+$ is called *plus*, and signifies addition. Thus, $a+b$ signifies that a is to be added to b.

15. The sign $-$ is called *minus*, and signifies subtraction. Thus $a-b$ signifies that b is to be subtracted from a.

16. Sometimes both the signs $+$ and $-$ occur before the same quantity, as $a\pm x$, in which case they signify that the quantity may be either added or subtracted, or that it is doubtful which operation is to be performed.

17. The sign $\times$ signifies multiplication. Thus, $a\times b$ denotes that a is to be multiplied by b; and $a\times b\times c\times d$, that the quantities a, b, c, and d, are to be multiplied together. This sign is read *into*. Thus, $a\times b$ is to be read, a into b. Sometimes a single point is substituted for $\times$. Thus $a.b$ signifies that a is multiplied by b.

18. The sign $\div$ signifies division. Thus, $a\div b$ signifies that a is to be divided by b.

19. Division is also represented by placing the divisor under the dividend, in the form of a fraction. Thus, $\frac{a}{b}$ signifies that a is to be divided by b; and $\frac{a-b}{a+b}$, that $a-b$ is to be divided by $a+b$.

20. The sign $>$, standing between two quantities, denotes that the one before it is greater than the one after it. Thus, $a>b$ signifies that the quantity a is greater than the quantity b.

21. On the other hand, the sign $<$ denotes that the quantity before it is less than the one after it. Thus, $b<a$ signifies that b is less than a.

22. The sign $.\dot{}.$ signifies *therefore*. Thus, $a=5 \therefore 3a=15$, is read, a is equal to 5, therefore $3a$ is equal to 15.

23. The signs $:\ ::\ :$ denote proportion. Thus, $a:b::c:d$ is to be read, as a is to b, so is c to d; and the signs, placed in their order, indicate that a has the same ratio to b that c has to d.

24. The sign $\sqrt{}$, called the radical sign, signifies the square root of the quantity which follows it; or, that the root of the quantity is to be extracted. Thus, $\sqrt{a}$ denotes that the square root of a is to be extracted.

25. By placing a figure above the sign, thus, $\sqrt[3]{}$, it is made the radical sign of any root whatever. Thus, $\sqrt[3]{a}$ signifies the cube root of a; $\sqrt[4]{a}$, the fourth root of a; $\sqrt[5]{a}$, the fifth root; $\sqrt[m]{a}$, the mth root, &c.

26. The power of a quantity is denoted by a figure placed above it at the right. Thus, a^2 signifies the second power of a; a^3 the third power of a; a^4 the fourth power, &c.

27. In operating with unknown quantities, it is frequently necessary to express the root of a certain power of a quantity; as, for instance, the 4th root of the 3d power of a. In this case, a fraction is to be used; the numerator denoting the power to which the quantity is to be raised, and the denominator the root of the power. Thus, $a^{\frac{2}{3}}$ denotes the cube root of the second power of a; $b^{\frac{6}{4}}$, the fourth root of the sixth power of b. By inverting the fraction, and writing it before the radical sign, we may represent the same. Thus, $\sqrt[\frac{3}{2}]{a}$, $\sqrt[\frac{4}{6}]{b}$, $=a^{\frac{2}{3}}$, $b^{\frac{6}{4}}$.

28. When a quantity is represented by a single letter or numeral, or several letters, placed one after another without the sign + or — between them, it is called a *simple quantity.* Thus, a, bc, cde, $3ab$, are simple quantities.

29. When a number of simple quantities are connected by the signs + or —, the result is a *compound quantity.* Thus, $a+b$, $be+cd$, $4a+5cd-x$, are compound quantities.

30. A *term* is a single letter or numeral, or several letters or numerals, which are not separated by the sign + or —. Thus, in the compound quantity $a+b$, a and b are the terms. So in $xy-y+z$, xy, y and z, are the terms.

31. When two or more members of a compound quantity are to be subjected to the same operation, in which they are to be regarded as one whole, they are connected by a line drawn over them, called a *vinculum,* or by enclosing them in a parenthesis. Thus, when we are to multiply $a+b+c$ by any number, as 3, we write $\overline{a+b+c}\times 3$, or $(a+b+c)\times 3$, or, more simply, $3(a+b+c)$. So $\overline{x+y}\times\overline{y+z}$, or $(x+y)\ (y+z)$ signifies that $x+y$ is to be regarded as a whole, and multiplied by $y+z$, taken also as a whole; whereas, if the line or parenthesis were not employed, $a+b\times 3$ would denote that b only is to be multiplied by 3, and the result would be $a+3b$.

32. When two or more quantities are multiplied together, each quantity is called a *factor*. Thus in ab, a and b are called factors; so, in cde, c, d and e, are severally called factors.

33. A *composite* number is one which is produced by the multiplication of two or more quantities or factors into each other. Thus, the quantity abc is a composite one, the factors of which are a, b, and c.

34. Quantities, which have the sign $+$ before them, either expressed or implied, are called *positive* or *affirmative* quantities. Thus $+a$, $+b$, or a, b, are positive or affirmative, the sign $+$ being always implied before a quantity which has no express sign prefixed.

35. Quantities, which have the sign $-$ prefixed are called *negative* quantities. Thus, $-a$, $-b$, -3, are negative quantities.

36. Where the signs are all positive or all negative, they are called *like* signs.

37. When some of the signs are positive and others negative, they are said to be *unlike*.

38. When a quantity consists of one term it is called a *monomial*, as a, ab, $3xy$, being the same as a simple quantity.

39. When a quantity consists of two terms it is called a *binomial*. Thus $a+b$, $x+y$, are called binomial quantities, and $a-b$ a *residual* binomial.

40. When a quantity consists of three terms it is called a *trinomial*. Thus, $a+b+c$, and $x+y+z$, are trinomials.

41. When a quantity consists of any number of terms greater than three it is called a *polynomial*. Thus, $a+b+c+d$, and $w+x+y+z$, are polynomials.

42. The *power* of a quantity is its square, cube, biquadrate, &c., called, also, its second, third, fourth power, &c.; as a^2, a^3, a^4, &c.

43. The *index* or *exponent* of a quantity is the number which denotes its power or root.

Thus, -1 is the index or exponent of a^{-1}; 2 is the index of a^2; $\frac{1}{2}$, of $a^{\frac{1}{2}}$, or $\sqrt{a}$; and m and $\frac{1}{n}$, of a^m and $a^{\frac{1}{n}}$.

44. When a quantity appears without any index or exponent, it is always understood to have for it unity, or 1.

Thus, a is the same as a^1, $2x$ is the same as $2x^1$; the 1 in such cases being usually omitted.

45. A *rational quantity* is that which can be expressed in finite terms, or without any radical sign or fractional index; as a, or $\frac{2a}{3}$, or $5a^2$, &c.

46. An *irrational quantity* is that which has no exact root, or which can only be expressed by means of the radical sign, or a fractional index; as $\sqrt{a}$, or $2^{\frac{1}{2}}$, $\sqrt[3]{a^2}$, or $a^{\frac{2}{3}}$, &c.

47. A square or cube number, &c., is that which has an exact square or cube root, &c.

Thus, 4 and $\frac{9}{16}a^2$ are square numbers; and 64 and $\frac{8}{27}a^3$ are cube numbers, &c.

48. A measure or divisor of any quantity is that which is contained in it some exact number of times, without a remainder.

Thus, 3 is the measure or divisor of 6, $7a$ is a measure of $35a$, and $9ab$ of $27a^2b^2$.

49. *Commensurable numbers* or quantities are such as have a common measure or divisor, or that can be each divided by the same quantity without a remainder.

Thus, 6 and 8, $2\sqrt{2}$ and $3\sqrt{2}$, $5a^2b$ and $7a^2b$, are commensurable quantities; the common divisors being 2, $\sqrt{2}$, and a^2b.

50. A *prime number* is that which has no exact divisor, except itself and unity; as 1, 2, 3, 5, 7, 11, 13, 17, &c.; and the intervening numbers, 4, 6, 8, 9, 10, 12, 14, and 16, are *composite numbers*.

51. Two or more numbers are said to be *incommensurable*, or *prime to each other*, when they have no common divisor except unity; as, 2 and 3, 5 and 7, 17 and 19, &c.

52. One quantity is called the *multiple* of another, when the former contains the latter a certain number of times without a remainder.

Thus, $15a$ is a multiple of $5a$, and $6a$ of $3a$.

53. The *reciprocal* of any quantity is unity divided by that quantity.

The reciprocal of any fraction is that fraction inverted.

Thus, the reciprocal of a or $\frac{a}{1}$ is $\frac{1}{a}$; the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, and the reciprocal of $\frac{a+b}{a-b}$ is $\frac{a-b}{a+b}$.

54. A *series* is a rank or succession of quantities, which usually proceed according to some certain law; as $1+\frac{1}{2}a+\frac{1}{4}a^2+\frac{1}{8}a^3+\frac{1}{16}a^4$, &c.

PRACTICAL EXAMPLES.

55. In calculating the numerical values of the following Algebraic Expressions, let $a=6$, $b=5$, $c=4$, $d=1$, and $e=0$.

1. $a^2+2ab-c+d=36+60-4+1=93.$
2. $2a^3-3a^2b+c^3=432-540+64=-44.$
3. $a^2\times(a+b)-2abc=36\times11-240=156.$
4. $2a\sqrt{b^2-ac}+\sqrt{2ac+c^2}=12\times1+8=20.$
5. $3a\sqrt{2ac+c^2}$, or $3a(2ac+c^2)^{\frac{1}{2}}=18\sqrt{64}=144.$
6. $\sqrt{(2a^2-\sqrt{6ac+e^2})}=\sqrt{(72-\sqrt{144})}=\sqrt{60}.$
7. $\sqrt{(a^2b+4b^2-5\sqrt{cd})}=\sqrt{(180+100-10)}=\sqrt{270}.$
8. $\sqrt{(ab^2+2b^3-5\sqrt{dc})}=\sqrt{(150+250-10)}=\sqrt{390}.$
9. $\frac{2a+3c}{6d+4e}+\frac{4bc}{\sqrt{2ac+c^2}}=\frac{24}{6}+\frac{80}{\sqrt{64}}=\frac{24}{6}+\frac{80}{8}=4+10=14.$
10. $\frac{2ab-5c}{4b-10d}+\left(\frac{3a-\frac{1}{2}bc}{5a-12d}\right)^{\frac{1}{2}}=\frac{40}{10}+\sqrt{\frac{18-10}{30-12}}=\frac{40}{10}+\sqrt{\frac{4}{9}}=$ $4\frac{2}{3}.$
11. $2a^2+3bc-5=127.$

12. $5a^2b-10ab^2+27e=-600.$

13. $7a^2+(b-c)\times(d-e)=253.$

14. $\frac{ab^2}{c}\times d-\frac{a-b}{d}+27a^2e=36\frac{1}{2}.$

15. $3\sqrt{}\,c+2a\sqrt{}\,(2a+b-d)=54.$

16. $a\sqrt{}\,(a^2+e)+3bc(a^2-b^2)=696.$

17. $3a^2b+\sqrt[3]{}\,(c^2-\sqrt{2ac+c^4})-3e=542.$

18. $\frac{2b+c}{3a-c}-\frac{\sqrt{}\,5b+3\sqrt{}\,c+d}{2a+c}=\frac{1}{4}.$

SECTION II.

ADDITION.

Art. 56. Addition in Algebra is the connecting together of several quantities by their appropriate signs.

57. The operation consists in collecting into one term all the like quantities, and so arranging the several terms, thus obtained, as by signs to indicate the proper sum of all the quantities, both like and unlike.

58. Addition in Algebra embraces three cases.

I. When the quantities are alike, and their signs alike also; as, a, $3a$; or, $-b$, $-4b$.

II. When the quantities are alike, and their signs unlike; as, $3b$, $-5b$.

III. When the quantities are unlike, some having like and others unlike signs; as, $3a$, $4b$, $-4x$.

Case I.

59. When the quantities are alike, and their signs alike.

Rule. *Add together the coefficients belonging to the like quantities, and place their sum before the common letter* *or letters, with the common sign prefixed; and the result will be the sum required.*

Thus, let it be required to add together $3ab$, $7ab$, $8ab$, the operation will be as follows:—

$$\begin{array}{r} 3ab \\ 7ab \\ 8ab \\ \hline 18ab. \end{array}$$

The reason of this rule is obvious; for, since ab, whatever be its value, must represent the same quantity in every instance, it is evident that 3 times, 7 times, and 8 times the same quantity, will make 18 times the same.

In like manner, let it be required to add together $-7b$, $-5b$, and $-6b$.

$$\begin{array}{r} -7b \\ -5b \\ -6b \\ \hline -18b. \end{array}$$

EXAMPLES.

(1)	(2)	(3)	(4)	(5)
$3a$	$7h$	$-3ax$	$4xy^2$	$3a-2y^2$
$4a$	$5h$	$-4ax$	xy^2	$4a-3y^2$
$6a$	$3h$	$-ax$	$3xy^2$	$6a-y^2$
a	$8h$	$-2ax$	$7xy^2$	$a-6y^2$
$5a$	h	$-7ax$	$2xy^2$	$5a-2y^2$
$19a$	$24h$	$-17ax$	$17xy^2$	$19a-14y^2$

(6)	(7)	(8)	(9)	(10)
$7x$	$14abc$	$5y$	$-4mn$	$5h+x$
$4x$	$11abc$	y	$-3mn$	$h+2x$
$11x$	$5abc$	y	$-mn$	$2h+4x$
$9x$	$4abc$	$3y$	$-11mn$	$h+x$
x	abc	$4y$	$-mn$	$7h+6x$

11. Add $7a$, $11a$, a, $4a$, $6a$, and $3a$ together. *Ans.* $32a$.

12. Add $4h$, $6h$, h, h, $11h$, and $7h$ together. *Ans.* $30h$.

13. Add together $(3a^2-b)$, $(7a^2-4b)$, and (a^2-b).
Ans. $11a^2-6b$.

14. What is the sum of $3\sqrt{a^2}$, $4\sqrt{a^2}$, $\sqrt{a^2}$, $7\sqrt{a^2}$, and $2\sqrt{a^2}$? *Ans.* $17\sqrt{a^2}$.

15. Add together $3\sqrt{a+b}$, $6\sqrt{a+b}$, $\sqrt{a+b}$, and $12\sqrt{a+b}$.
Ans. $22\sqrt{a+b}$.

CASE II.

60. When the quantities are alike, and have unlike signs.

RULE. *Add all the affirmative coefficients into one sum, and those that are negative into another; then subtract the less of these results from the greater, and prefix the sign of the greater to the difference, annexing the common letter or letters.*

Required the sum of $+7ax$, $-4ax$, $-3ax$, $+17ax$, $-ax$, and $+ax$.

$$\begin{array}{r} 7ax \\ -\ 4ax \\ -\ 3ax \\ 17ax \\ -\ ax \\ ax \\ \hline 17ax. \end{array}$$

We find the sum of the plus quantities to be $25ax$, and the sum of the negative quantities $-8ax$; and the difference between those coefficients is 17, which we prefix to ax; thus, $17ax$.

The reason of this Rule is obvious, when we consider that two equal quantities, the one with a positive and the other with a negative sign, exactly cancel each other, so that their sum is nothing. Of course, then, when two like quantities of opposite signs are not equal, the difference between them must be the proper sum, which will be positive or negative according to the affirmative or negative character of the larger quantity.

EXAMPLES.

(1)	(2)	(3)	(4)	(5)
$7a$	$-6m$	$4ax$	$13n$	$7a-mp^2$
$-3a$	m	$-3ax$	n	$a+6mp^2$
a	$-11m$	ax	$-20n$	$-11a-3mp^2$
$-5a$	$5m$	$-7ax$	$6n$	$8a+11mp^2$
$11a$	$-m$	ax	$8n$	$-9a-7mp^2$
a	$20m$	$12ax$	$-n$	$18a-15mp^2$
$12a$	$8m$	$8ax$	$7n$	$14a-9mp^2$

(6)	(7)	(8)	(9)	(10)
$6y$	$4mn$	$-8xy$	$-4pn$	$8xy-m^2p$
$-7y$	mn	$7xy$	pn	$3xy+m^2p$
$4y$	$3mn$	$-4xy$	pn	$-11xy-18m^2p$
$-11y$	$18mn$	$-xy$	$-11pn$	$-4xy+9m^2p$
$9y$	$7mn$	$9xy$	$7pn$	$-8xy-3m^2p$
$-2y$	$8mn$	$-3xy$	pn	$12xy+12m^2p$

11. Add $4+a^2x$, $6-a^2x$, $3+6a^2x$, $15-5a^2x$, $3+a^2x$, and $6+7a^2x$ together. *Ans.* $37+9a^2x$.

12. Add $14ax-6y$, $7ax+y$, $5ax-7y$, $9ax-11y$, and $8ax+3y$ together. *Ans.* $43ax-20y$.

13. Add $3a-4b+6c$, $7a+11b-3c$, $8a+b-7c$, and $a-11b+15c$ together. *Ans.* $19a-3b+11c$.

14. Add $16x^2-5y^3-16$, $3x^2+4y^3-5$, x^2+3y^3-37, x^2-y^3+7, $6x^2+7y^3-11$, and $2x^2-3y^3-21$ together.
Ans. $29x^2+5y^3-83$.

15. Add $5a-b$, $3b+3c$, $4a-5c$, $5a-5b-c$, $7a-6c$, and $11a+4b-7c$ together. *Ans.* $32a+b-16c$.

CASE III.

61. When the quantities are unlike, some having like and others unlike signs.

It is evident that unlike quantities cannot be united into one; or otherwise added than by means of their signs.

Thus, for example, if a be supposed to represent 20, and b 12, then the sum of a and b can be neither twice 20 nor twice 12, but it must be $20+12=32$, that is, $a+b$.

62. Hence, to add unlike quantities, we have the following

Rule. *Collect all the like quantities together, as in* Case II., *and write down those that are unlike, one after another, with their proper signs.*

63. When several quantities are to be added together, it is immaterial in what order they are written.

Thus, $a+b-c$, $a-c+b$, $-c+a+b$, are equivalent expressions.

EXAMPLES.

(1)

$$\begin{array}{r}
3ax \\
4mn \\
-6y^2 \\
7xy \\
\hline
3ax+4mn-6y^2+7xy.
\end{array}$$

(2)

$$\begin{array}{l}
7\ a+7m \\
6\ a-7x \\
4xy-5m \\
3x\ +8xy \\
\hline
13a+12xy+2m-4x.
\end{array}$$

(3)

$$\begin{array}{r}
9x^2y \\
-7x^2y \\
3axy \\
-4xy^2 \\
\hline
\end{array}$$

(4)

$$\begin{array}{l}
14ax - 2x^2 \\
5ax^2 + 3xy \\
8y^2 - 4ax \\
3x^2 +26 \\
\hline
\end{array}$$

(5)

$$\begin{array}{l}
4ax-130+3x^{\frac{1}{2}} \\
5a^2 +3ax+9x^2 \\
7xy-4x\tfrac{1}{2}+9^0 \\
\sqrt{x}-40\ -6x^2 \\
\hline
\end{array}$$

6. Add together $a+x$ and $y-c$. *Ans.* $a-c+x+y$.

7. Add $3a+b-10$, $c-d-a$, $-4c+2a-3b-7$, and $4x^2+5-18m$ together. *Ans.* $4a-2b-12-3c-d+4x^2-18m$.

8. Add $7a-5y^3$, $8\sqrt{x}+2a$, $5y^3-\sqrt{x}$, and $-9a+7\sqrt{x}$ together. *Ans.* $14\sqrt{x}$.

9. Add $4mn+3ab-4c$, $3x-4ab+2mn$, and $3m^2-4p$ together. *Ans.* $6\ mn-ab-4c+3x+3m^2-4p$.

10. Find the sum of $3a^2+2ab+4b^2$, $5a^2-8ab+b^2$, $-a^2+5ab-b^2$, $18a^2-20ab-19b^2$, and $14a^2-3ab+20b^2$.

Ans. $39a^2-24ab+5b^2$.

11. Find the sum of $4x^3-5a^3-5ax^2+6a^2x$, $6a^3+3x^3+4ax^2+2a^2x-17x^3+19ax^2-15a^2x$, $13ax^2-27a^2x+18a^3$, $3a^2x-20a^3+12x^3$, and $31a^2x-2x^3-31ax^2-7x^3$. *Ans.* $-7x^3-a^3$.

64. Coefficients, whether figures or letters, that are common to several terms, may be connected with them by a parenthesis.

	(12)	(13)
Add	$amx+2dy$	$hy+4mx$
	$2cx-3dy$	$my+3dx$
	$3ax+5y$	$4ny-9mx$
	$(am+2c+3a)x+(5-d)y.$	$(h+m+4n)y+(3d-5m)x.$

14. Add $4ax-5my$, $3dx+7ny$, and $7mx+4my$, together. *Ans.* $(4a+3d+7m)x+(7n-m)y$.

15. Add $3hz-5x$, $4mz+nx$, and $5az-4px$, together. *Ans.* $(3h+4m+5a)z+(n-5-4p)x$.

SECTION III.

SUBTRACTION.

ART. **65.** Subtraction is the taking of one quantity from another, or the method of finding the difference between any two quantities or sets of quantities of the same kind.

66. If it be required to subtract $10-7$ from 12, we might first subtract 7 from $10=3$, and take the 3 from $12=9$; or we might take the 10 from 12, and the remainder 2 must necessarily be increased by 7 to produce the correct result.

If from a we wish to subtract $c-d$, we first subtract c, and it gives $a-c$. This quantity, since we have taken d too much from a, is too small by d. Therefore d must be added, thus, $a-c+d$.

67. If a simple quantity is to be taken from another simple quantity, it is only necessary to write them one after the other; thus, if 8 is to be taken from 15, it may be expressed thus, $15-8=7$.

If it were required to subtract b from a, it should be written thus, $a-b$; but if we were to subtract $a-b$ from $c+d$, it is evident that if only a were to be taken, it would be written thus, $c+d-a$. But this evidently gives a result too small; for a was to be lessened by b before the subtraction. Therefore, as the remainder is too small by b, we must add b to the remainder, which will give $c+d-a+b$; for it makes no difference in the result whether the minuend be increased or the subtrahend lessened.

Subtract $7-4$ from 13. Taking 7 from 13 leaves 6; but 6 is too small, for the 7 should have been lessened by 4, and we must either subtract the 4 from the 7 before the operation, or add it to the remainder; and, if added to the remainder, the expression will be thus, $6+4=10$.

68. We therefore see the propriety of the following

RULE. *Change the signs of all the quantities to be subtracted, and proceed as in Addition.*

SIMPLE QUANTITIES.

	(1)	(2)	(3)	(4)	(5)	(6)
From	$+7a$	$-16x$	$+17d$	$-29g$	$+15b$	$-6c$
Take	$+2a$	$-5x$	$+8d$	$-18g$	$+7b$	$-c$
	$+5a$	$-11x$	$+9d$	$-11g$	$+8b$	$-5c$

The above questions are performed as in Arithmetic, the minuend being the larger number, and having the same sign as the subtrahend.

	(7)	(8)	(9)	(10)	(11)	(12)	(13)
From	$-8a$	$+7x$	$+18y$	$-3b$	$-7c$	$+8d$	$-6h$
Take	$-15a$	$+14x$	$+20y$	$-7b$	$-15c$	$+11d$	$-8h$
	$+7a$	$-7x$	$-2y$	$+4b$	$+8c$	$-3d$	$+2h$

In these examples the minuend is taken from the subtrahend, and all the signs in the subtrahend are changed.

	(14)	(15)	(16)	(17)	(18)	(19)	(20)
From	$+27a$	$-6b$	$-7c$	$+8g$	$+11h$	$-5x$	$-7y$
Take	$-13a$	$+18b$	$-c$	$-9g$	$-15h$	$+17x$	$-15y$
	$+40a$	$-24b$	$-6c$	$17g$	$+26h$	$-22x$	$+8y$

In these examples we change, *mentally*, all the signs in the subtrahend, and then proceed as in addition. These questions may all be proved, as in Arithmetic, by adding the remainder to the subtrahend.

COMPOUND QUANTITIES.

69. The same rule must be observed in subtracting compound quantities as in simple quantities; that is, all the signs of the quantities to be subtracted must be changed, the signs + to —, and the signs — to +; we then proceed as in addition.

	(1)	OPERATION.
From	$ab+cd-ax-7$	$ab+cd-ax-7$
Take	$4ab-3cd+4ax-15$	$-4ab+3cd-4ax+15$
		$-3ab+4cd-5ax+8$

70. It is a better way for the pupil to *conceive* the signs in the subtrahend changed, but to let them remain without alteration, otherwise it might be difficult to correct errors that might arise in the operation.

	(2)	(3)
From	$7x+5y-3a-6h$	$7abc-11x+5y-48$
Take	$x-7y+5a+11h$	$11abc+3x+7y+100$
	$6x+12y-8a-17h$	$-4abc-14x-2y-148$

	(4)	(5)
From	$14h-4z+9y+x$	$9x-5abc-6h-51$
Take	$-3h-7z+41y-17x$	$19x-7abc-8h+1$
	$17h+3z-32y+18x$	$-10x+2abc+2h-52$

	(6)	(7)
From	$3xy-a^6-3h^3-y^2$	$7x^2-a^3b^5+7y^5+8h^4$
Take	$-xy-a^6+7h^3-10y^2$	$x^2+a^3b^5-11y^5-h^4$
	$4xy-0-10h^3+9y^2$	$6x^2-2a^3b^5+18y^5+9h^4$

	(8)	(9)
From	$5\sqrt{ax^2}-3y^2+7a^{\frac{1}{2}}-1$	$8x^{\frac{2}{3}}+y^5+\sqrt{7h}+5$
Take	$3\sqrt{ax^2}+y^2-5a^{\frac{1}{2}}+7$	$4x^{\frac{2}{3}}-3y^5-\sqrt{7h}-6$
	$2\sqrt{ax^2}-4y^2+12a^{\frac{1}{2}}-8$	$4x^{\frac{2}{3}}+4y^5+2\sqrt{7h}+11$

10. From $3a-5b+6h-d$ take $a+b-7d$.

Ans. $2a-6b+6h+6d$.

11. From $31x^2-3y^2+ab$ take $17x^2+5y^2-4ab+7$.

Ans. $14x^2-8y^2+5ab-7$.

12. From $5f+14b-9d$ take $-3f+7b-15d$.

Ans. $8f+7b+6d$.

13. From $11a-7b+c$ take $a+7b-3c+11$.

Ans. $10a-14b+4c-11$.

14. From m^2+3n^3 take $-4m^2-6n^3+71x$.

Ans. $5m^2+9n^3-71x$.

15. From $31a-15x-7$ take $2a-25x+y^2$.

Ans. $29a+10x-y^2-7$.

16. From abc^2-xy^3 take $-6abc^2+3xy^3-7h$.

Ans. $7abc^2-4xy^3+7h$.

17. From $11ch^2-5$ take $5ch^2-5+47x$.

Ans. $6ch^2-47x$.

18. From mn^2+kt take $-7mn^2+48x-y^2$.

Ans. $8mn^2+kt-48x+y^2$.

19. From $47abh-37+96y^2$ take $7abh$.

Ans. $40abh-37+96y^2$.

20. Take $7x^2y^3+hm$ from $8x^2y^3+17$.

Ans. $x^2y^3-hm+17$.

21. Take $5b^2-3c+59m$ from $11b^2$.

Ans. $6b^2+3c-59m$.

22. Take $6a-3b-5c$ from $6a+3b-5c+1$.

Ans. $6b+1$.

23. Take $41x^2+7y^3+abc$ from m^2.
Ans. $m^2-41x^2-7y^3-abc$.

24. Take x^2 from $-17x^2+14y-a+b$.
Ans. $-18x^2+14y-a+b$.

25. Take $a-b$ from $a+b$. *Ans.* $+2b$.

26. From $9xz$ take $xz-7h-5m^3+7$.
Ans. $8xz+7h+5m^3-7$.

27. From $11hm+8n^2$ take x^2-y^2.
Ans. $11hm+8n^2-x^2+y^2$.

28. From $a+b$ take $a-b$, and $a-b$, and $-a+b$.
Ans. $+2b$.

29. From $a-b-c$ take $-a+b+c$ and $a-b+c$.
Ans. $a-b-3c$.

71. When similar quantities that are to be subtracted have literal coefficients, the operation may be performed by placing the coefficients with their proper signs within a parenthesis, and then subjoining the common quantity; thus,

From	$ay-h$	From	ax^3+gy^2
Take	$dy-c$	Take	bx^3-hy^2
	$(a-d)y+c-h$		$(a-b)x^3+(g+h)y^2$.

72. If a set of quantities enclosed in a parenthesis is combined with others by means of the sign $+$, the parenthesis can have no effect upon the result, and may, therefore, be retained or not, at pleasure.

Thus, $a+(b+c)$ is evidently equivalent to $a+b+c$; for it can make no difference whether b and c be first added together, and their sum then be added to a, or the sum of the three quantities, a, b, c, be taken at once.

Again, $x-y+(b-z)$ will amount to the same thing as $x-y+b-z$; for it is immaterial whether $b-z$ be added to $x-y$ at once, or b be added to it first, and from the result z be subtracted.

The subtraction of a polynomial may be indicated without performing the operation, by inclosing the quantity to be subtracted in a parenthesis, and prefixing the sign $-$.

If we wish to subtract $7a-5x+6y$ from $11a-2x+8y$, it may be indicated thus $(11a-2x+8y)-(7a-5x+6y)$.

And $7a-3b+c+g-p$, taken from $10a$, leaves $10a-(7a-3b+c+g-p)$; being equivalent to $3a+3b-c-g+p$.

If, therefore, a quantity enclosed in a parenthesis be combined with another by means of the sign —, the rule laid down in Art. 68 shows that the signs of the terms of this quantity must be changed whenever the parenthesis is removed.

Thus, $a-(b+c)$ is equivalent to $a-b-c$; because it can be of no importance whether b be first subtracted from a, and c then be taken from the remainder, or the sum of b and c be subtracted from a at once.

Consequently, a parenthesis, with a negative sign preceding it, may be introduced into any compound algebraical expression, provided the signs of all the symbols comprised in it be changed.

Thus, $a-x-b+y$ is equivalent to $a-x-(b-y)$, or $a-(x+b-y)$, or $a+y-(b+x)$, or $y-(x+b-a)$.

Similar considerations will enable us to dispense with the use of parentheses, without altering the values of the expressions in which they are found, when one or more such parentheses are included within another.

Thus, $a-[b-(c+d)]$ is manifestly equivalent to $a-[b-c-d]$, which is also equivalent to $a-b+c+d$.

Also, $a-\{a+b-[a+b-c-(a-b+c)]\}=a-\{a+b-[a+b-c-a+b-c]\}=a-\{a+b-[2b-2c]\}=a-\{a+b-2b+2c\}=a-\{a-b+2c\}=a-a+b-2c=b-2c$.

EXAMPLES FOR PRACTICE.

1. What is the value of the expression $(1-2x+3x^2)+(3+2x-x^2)$?
Ans. $4+2x^2$.

2. Reduce to its simple form the expression $5a-4b+3c+(-3a+2b-c)$.
Ans. $2a-2b+2c$.

3. What is the value of the expression $(a-b-c)+(b+c-d)+(d-e+f)+(e-f-g)$?
Ans. $a-g$.

4. Exhibit $a-(b-c)+b-(a-c)+c-(a-b)$ in its simplest form. *Ans.* $-a+b+3c$.

5. From $3(x^2+y^2)$ take $[(x^2+2xy+y^2)-(2xy-x^2-y^2)]$. *Ans.* x^2+y^2.

6. From $6x^2+2y^2-(3x^2+y^2)$ take $2x^2+4y^2-(4x^2-y^2)$. *Ans.* $5x^2-4y^2$.

73. Algebra differs from Arithmetic in the use of negative quantities. In Algebra, every quantity is either positive or negative, according as it is affected with the sign plus or minus; and, as we have observed above, whenever a quantity has not either of these signs prefixed, the sign + is understood, and the quantity is said to be positive. Thus 5, or +5, is positive; but —5 is negative. Positive quantities are also called affirmatives. Some mathematicians, in treating this subject, have involved it in much perplexity, and, in our opinion, in absurdities, by considering —5, or $-a$, as quantities *less than nothing;* much to the injury, if not to the disgrace, of the science. But the student is to observe that —5 denotes just the same number and quantity as +5, but with the additional considerations, that the former is to be *subtracted*, while the latter is to be *added*.

The simplest illustration of positive and negative quantities may be derived from a merchant's credits and debts. Five dollars is the same sum, whether it be due to him, or he owe it to another; but, in one case it may be considered as *positive* \$5, for it is an addition to his property; and in the other as *negative* \$5, for it is subtracted from his property. And, if the sum of his debts exceeds the sum of his credits by \$1000, the state of his affairs may be represented by —\$1000; and, undoubtedly, he is worse than if he had nothing, and owed nothing. In such a case, indeed, the man is often said, in *mercantile language*, to be minus one thousand dollars. Whereas, if the sum of his credits exceeds the sum of his debts by \$1000, the state of his affairs may justly be represented by +\$1000. These opposite signs, then, without at all affecting the absolute magnitude of the quantities to which they are prefixed, intimate the additional consideration that those quantities are in *contrary* circumstances.

SECTION IV.

MULTIPLICATION.

ART. **74.** Multiplication is the repeating of a quantity as many times as there are units in another; it is virtually the same in Algebra as in Arithmetic.

75. The multiplicand and multiplier may be considered as factors; and, in all operations, either may be taken for the other.

Thus, if 6 be multiplied by 7, or a by b, the result is the same as if 7 be multiplied by 6, or b by a.

76. When several letters are written after one another, it implies that they are all multiplied together.

Thus, $abcd$ is the same as $a \times b \times c \times d$; and it is immaterial in what order they stand; for $abcd$, $cdab$, and $bdca$, are synonymous terms.

77. Multiplication is commonly divided into three cases.

I. When the multiplicand and multiplier are simple quantities.

II. When the multiplicand is a compound quantity, and the multiplier is a simple one.

III. When both the multiplicand and multiplier are compound quantities.

CASE I.

78. When the multiplicand and multiplier are simple quantities.

RULE. *Multiply the coefficients of both terms together, and to the product annex the letters in both factors, remembering that the product of like signs is plus, and of unlike signs is minus. That is, plus (+) multiplied by plus (+), and minus (—) multiplied by minus (—), give plus (+); and plus (+) multiplied by minus (—), and minus (—) multiplied by plus (+), give minus (—).*

ILLUSTRATION.

79. 1. If a plus quantity is multiplied by a plus quantity, the result will be a plus quantity. Thus,

If $+a$ is multiplied by $+b$, it is evident that $+a$ is to be repeated as many times as there are units in $+b$; that is, b times $a=+ab$.

2. If a minus quantity is multiplied by a plus quantity, or a plus quantity by a minus quantity, the result will be a minus quantity. Thus,

If $-c$ is to be multiplied by $+d$, it is evident that $-c$ must be repeated as many times as there are units in d; that is, d times $-c=-cd$. The result will be the same if $+c$ is multiplied by $-d$.

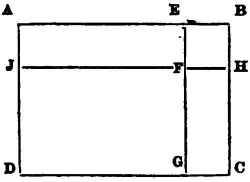
3. If a minus quantity be multiplied by a minus quantity, the result will be a plus quantity.

To illustrate this, let $a-b$ be multiplied by $c-d$.

The product of $a-b$ by c is $ac-bc$; but it is evident that this product is as many times too large as there are units in d. Therefore the product of $a-b$ by $d=ad-bd$, must be subtracted from $ac-bc$, thus $(ac-bc)-(ad-bd)=ac-bc-ad+bd$; but $+bd$ is the product of $-b$ and $-d$; therefore a minus quantity multiplied by a minus is a plus quantity, *Q.E.D.*

80. That the product of two minus quantities produces a plus, may be illustrated by the following diagram:

Let *ABCD* be a right-angled parallelogram. Let *JH* be parallel to *AB*, and *EG* parallel to *AD*. Then the figure will contain four right-angled parallelograms, *JFGD*, *AJFE*, *EBHF*, and *FHCG*. Let *AB*, which is equal to *JH*, $=a$, and *EB* or its equal $FH=b$; then AE, or its equal JF, will be $=a-b$. Also let $AD=c$, and $AJ=d$, then JD or $FG=c-d$. Now, to find the contents of $JFGD$, we must multiply the adjacent sides of the parallelogram together, which are JD and JF. But $JD=c-d$, and $JF=a-b$; therefore the contents of the parallelogram will be $(a-b)\times(c-d)=ac-ad-bc+bd$.



But ac is the contents of the figure $ABCD$, for it is the product of the adjacent sides AB and AD. And this exceeds the contents of the figure $JFGD$ by the three parallelograms $AJFE$, $EBHF$, and $FHCG$. But ad is the contents of the figure $ABHJ$, for the side $AB=a$, and $AJ=d$, and these are the adjacent sides of the parallelogram. And bc is the contents of the figure $EBCG$; for $EB=b$, and AD or $BC=c$, and therefore $bc=EBCG$, for it is the product of the adjacent sides EB and BC. But the parallelograms $ABHJ$ and $EBCG$ both include the parallelogram $EBHF$; whereas it should be included by only one of them. It must, therefore, be returned. The contents of this figure $EBHF=bd$; for $FH=b$, and $HB=d$, and their product is bd. And as BF has been taken twice from the figure, it is restored by considering bd a plus quantity, thus $+bd$, *Q.E.D.*

EXAMPLES.

1. Multiply $4m$ by $3n$. *Ans.* $12mn$.
2. Multiply $3ab$ by $-5cd$. *Ans.* $-15abcd$.
3. Multiply $8mn$ by $4xy$. *Ans.* $32mnxy$.
4. Multiply $7pg$ by y. *Ans.* $7pgy$.
5. Multiply $-13adef$ by $6mnp$. *Ans.* $-78adefmnp$.
6. Multiply $7hp$ by $4tuz$. *Ans.* $28hptuz$.
7. Multiply $19ab$ by $-xyz$. *Ans.* $-19abxyz$.
8. Multiply $7an$ by $-2an$. *Ans.* $-14aann$.
9. Multiply $5aaa$ by $3aaa$. *Ans.* $15aaaaaa$.

10. Multiply $-4xy$ by $-xxyy$.
11. Multiply $-11dc$ by $-8dee$.
12. Multiply $9mn$ by $2mmn$.
13. Multiply $17abc$ by $-8abc$.
14. Multiply $11xyyy$ by $-yy$.
15. Multiply $-9mmm$ by $-nnn$.
16. Multiply pqt by pqt.

81. When the same letter is repeated in the product, for the sake of brevity one letter only need be written, with a figure placed after and above it, denoting the number of times the letter is taken as a factor.

This figure is called the *exponent* or power of the letter, and it shows how many times the letter is used as a factor. Thus, $a^3=a\times a\times a=aaa$, $4m^2=4\times m\times m=4mm$.

82. If two or more letters of the same kind, having exponents, are to be multiplied together, we write the letter, and place over it the sum of the exponents. Thus, the product of a^3 by $a^2=aaa\times aa=aaaaa=a^5$. Hence the following

RULE. *Add the exponents of the same letter, and place their sum over the product of the letter multiplied by the coefficients.*

17. Multiply $4m^4$ by $3m^2$.

$$4\times 3\times m^4\times m^2=12m^{4+2}=12m^6.$$

18. Multiply $-5n^3$ by $-4n^5$. *Ans.* $20n^8$.
19. Multiply $-3a^m$ by $3a^m$. *Ans.* $-9a^{2m}$.
20. Multiply $2x^m$ by $4x^n$. *Ans.* $8x^{m+n}$.
21. Multiply $3x^2b^3$ by $5a^3b$. *Ans.* $15a^5b^4$.
22. Multiply ab^2 by a^3b. *Ans.* a^4b^3.
23. Multiply a^3b^2c by a^2bd. *Ans.* a^5b^3cd.
24. Multiply $7a^5c^3$ by a^4cm. *Ans.* $7a^9c^4m$.
25. Multiply $9a^5b^3x^7$ by $-a^8b^2cx^4$. *Ans.* $-9a^{13}b^5cx^{11}$.
26. Multiply $15m^5n^6$ by $3mn$. *Ans.* $45m^6n^7$.
27. Multiply $3a^mb^n$ by $2a^mb^3$. *Ans.* $6a^{2m}b^{n+3}$.
28. Multiply $4x^my^n$ by $-x^ny^nz^5$. *Ans.* $-4x^{m+n}y^{2n}z^5$.

29. Multiply $17a^4c^2$ by $4aacc$. *Ans.* $68a^6c^4$.
30. Multiply $3a^{m+n}$ by $-4a^{m-n}$. *Ans.* $-12a^{2m}$.
31. Multiply $7a^m$ by $3a^{-m}$. *Ans.* 21.
32. Multiply $11n^2$ by $-5n^6$. *Ans.* $-55n^8$.
33. Multiply $4a^6$ by $-3a^{-2}$. *Ans.* $-12a^4$.
34. Multiply $7m^n$ by $3m^n$. *Ans.* $21m^{2n}$.
35. Multiply $6ab^2$ by a^3b^{-4}. *Ans.* $6a^4b^{-2}$.
36. Multiply a^{-6} by a^{-3}. *Ans.* a^{-9}.
37. Multiply x^{-n} by x^n. *Ans.* 1.
38. Multiply m^3 by m^{-3}. *Ans.* 1.

CASE II.

83. When the multiplicand is a compound quantity, and the multiplier is a simple quantity.

RULE. *Multiply each term of the multiplicand separately by the multiplier, and prefix the proper sign to each term of the product.*

EXAMPLES.

	(1)	(2)	(3)	(4)
Multiply	$3a+5x$	$7m-4n$	$3b-4c$	$5x+7b$
By	$4m$	$3a$	$5e$	$3m$
	$12am+20mx.$	$21am-12an.$	$15be-20ce.$	$15mx+21bm.$

(5)	(6)	(7)	(8)
$4x^2-3ax^2$	$4m^3+2n$	$8a^2bc-d$	$abc+m^n$
$3x$	$3m^2$	$5ad^2$	$4am$
$12x^3-9ax^3.$	$12m^5+6m^2n.$	$40a^3bcd^2-5ad^3.$	$4a^2bcm+4am^{n+1}.$

9. Multiply $5a^2x-7y+4x^3-3b^3$ by $4ay^2$.
Ans. $20a^3xy^2-28ay^3+16ax^3y^2-12ab^3y^2$.

10. Multiply $7a^2b^3+4am^2-6y$ by $4a^5m^3$.
Ans. $28a^7b^3m^3+16a^6m^5-24a^5m^3y$.

11. Multiply $4a^2b^3-6a^3c+c^2$ by $-5a^2$.
Ans. $-20a^4b^3+30a^5c-5a^2c^2$.

12. Multiply $-ab^2-3x^3-14m^{-2}$ by $-am$.

Ans. $a^2b^2m+3amx^3+14am^{-1}$.

CASE III.

84. When both the multiplicand and multiplier are compound quantities.

RULE. *Multiply each term of the multiplicand by each term of the multiplier, remembering that the product of like signs is* $+$, *and the product of unlike signs is* $-$; *then add together all the products.*

NOTE. Terms which are alike should be placed under one another.

EXAMPLES.

(1)

$$\begin{array}{ll}
\text{Multiply} & 3a+4b \\
\text{By} & 2a+\ b \\
\hline
 & 6a^2+8ab \\
 & \qquad\ +3ab+4b^2 \\
\hline
 & 6a^2+11ab+4b^2.
\end{array}$$

(2)

$$\begin{array}{l}
x+y \\
2x-y \\
\hline
2x^2+2xy \\
\qquad\ -xy-y^2 \\
\hline
2x^2+xy-y^2
\end{array}$$

(3)

$$\begin{array}{l}
7ax-4y+6m \\
4ay+2y \\
\hline
28a^2xy-16ay^2+24amy \\
\qquad\qquad +14axy-8y^2+12my \\
\hline
28a^2xy-16ay^2+14axy-8y^2+24amy+12my.
\end{array}$$

(4)

$$\begin{array}{l}
2x^2+y \\
x^2+y \\
\hline
2x^4+x^2y \\
\qquad +2x^2y+y^2 \\
\hline
2x^4+3x^2y+y^2.
\end{array}$$

(5)

$$\begin{array}{l}
3a+4m \\
2a-2m \\
\hline
6a^2+8am \\
\qquad -6am-8m^2 \\
\hline
6a^2+2am-8m^2.
\end{array}$$

(6)

$$\begin{array}{l}
3a-2b \\
2a-5b \\
\hline
6a^2-4ab \\
\qquad -15ab+10b^2. \\
\hline
6a^2-19ab+10b^2.
\end{array}$$

(7)

$$\begin{array}{rrrrrr}
4a^3 & -5a^2b & -8ab^2 & +2b^3 & & \\
2a^2 & -3ab & -4b^2 & & & \\
\hline
8a^5 & -10a^4b & -16a^3b^2 & +4a^2b^3 & & \\
 & -12a^4b & +15a^3b^2 & +24a^2b^3 & -6ab^4 & \\
 & & -16a^3b^2 & +20a^2b^3 & +32ab^4 & -8b^5 \\
\hline
8a^5 & -22a^4b & -17a^3b^2 & +48a^2b^3 & +26ab^4 & -8b^5.
\end{array}$$

85. When positive and negative terms balance each other in the product, they should be cancelled.

(8)

$$\begin{array}{rrrr}
a^2 & +ax & +x^2 & \\
a & -x & & \\
\hline
a^3 & +a^2x & +ax^2 & \\
 & -a^2x & -ax^2 & -x^3 \\
\hline
a^3 & & & -x^3.
\end{array}$$

(9)

$$\begin{array}{rrrrr}
1 & -x & +x^2 & -x^3 & \\
1 & +x & & & \\
\hline
1 & -x & +x^2 & -x^3 & \\
 & +x & -x^2 & +x^3 & -x^4 \\
\hline
1 & & & & -x^4.
\end{array}$$

86. The continued product of factors is often expressed in one line.

10. $(1+x)\ (1+x^4)\ (1-x+x^2-x^3)=1-x^8$.

11. $(a+2x)\ (a-3x)\ (a+4x)=a^3+3a^2x-10ax^2-24x^3$.

12. Required the continued product of $3a-x$, $2a+4x$, and $4a-2x$. *Ans.* $24a^3+28a^2x-36ax^2+8x^3$.

13. Multiply $3x^2-2xy-y^2$ by $2x-4y$. *Ans.* $6x^3-16x^2y+6xy^2+4y^3$.

14. Multiply x^2+2x+1 by x^2-2x+3. *Ans.* x^4+4x+3.

15. Multiply $a+b-c$ by $a-b+c$. *Ans.* $a^2-b^2+2bc-c^2$.

16. Multiply $3a-2b$ by $-2a+4b$. *Ans.* $-6a^2+16ab-8b^2$.

17. Multiply $5a^2-3ab+4b^2$ by $6a-5b$. *Ans.* $30a^3-43a^2b+39ab^2-20b^3$.

18. Multiply a^2+ab+b^2 by $a-b$. *Ans.* a^3-b^3.

19. Multiply a^4-x^4 by a^4-x^4. *Ans.* $a^8-2a^4x^4+x^8$.

20. Multiply $2x^2-3xy+6$ by $3x^2+3xy-5$.
Ans. $6x^4-3x^3y+8x^2-9x^2y^2+33xy-30$.

21. Multiply $5a^2-4ax+3x^2$ by $2a^2-3ax-4x^2$.
Ans. $10a^4-23a^3x-2a^2x^2+7ax^3-12x^4$.

22. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$.
Ans. $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$.

23. Multiply a^3-3a^2+3a-1 by a^2-2a+1.
Ans. $a^5-5a^4+10a^3-10a^2+5a-1$.

24. Multiply a^m-a^n by $2a-a^n$.
Ans. $2a^{m+1}-2a^{n+1}-a^{m+n}+a^{2n}$.

25. Multiply $a^4-a^3x+a^2x^2-ax^3+x^4$ by $a+x$.
Ans. a^5+x^5.

MULTIPLICATION BY DETACHED COEFFICIENTS.

87. The coefficients of the polynomials should be arranged according to the successive powers of the letters, increasing or decreasing by a common difference; and, when this common difference is wanting, its place should be supplied by zero.

The following examples will illustrate the above :

1. Multiply a^2+2a+1 by a^2-2a+1.

$$\begin{array}{lllll} 1 & +2 & +1 & & \\ 1 & -2 & +1 & & \\ \hline 1 & +2 & +1 & & \\ & -2 & -4 & -2 & \\ & & +1 & +2 & +1 \\ \hline 1 & +0 & -2 & +0 & +1. \end{array}$$

In adding the coefficients of the partial products, we perceive that the second and fourth places are a zero : but the letters must be written with their powers regularly ascending from left to right ; and, where zero is the coefficient, the value of the quantity is nothing. Thus, $a^4+0a^3-2a^2+0a+1=a^4-2a^2+1$, because zero is the coefficient of the second and fourth terms.

2. Multiply x^4-x^2 by x^3+x.

$$\begin{array}{l}
1+0-1 \\
1+0+1 \\
\hline
1+0-1 \\
+1+0-1 \\
\hline
1+0+0+0-1.
\end{array}$$

With the letters and their powers added, it will be

$$x^7+0x^6+0x^5+0x^4-x^3=x^7-x^3.$$

The second, third, and fourth terms are of no value.

3. Multiply $3a^3-4ab^2+6b^3$ by $2a^2-4b^2$.

$$\begin{array}{l}
3+0-4+6 \\
2+0-4 \\
\hline
6+0-8+12 \\
-12-0+16-24 \\
\hline
6+0-20+12+16-24.
\end{array}$$

We now annex the letters with their proper powers, decreasing by a constant common difference, thus:

$$6a^5+0a^4b-20a^3b^2+12a^2b^3+16ab^4-24b^5=$$
$$6a^5-20a^3b^2+12a^2b^3+16ab^4-24b^5.$$

4. Multiply $2a^3-3ab^2+5b^3$ by $2a^2-5b^2$.

$$\begin{array}{l}
2+0-3+5 \\
2+0-5 \\
\hline
4+0-6+10 \\
-10-0+15-25 \\
\hline
4+0-16+10+15-25.
\end{array}$$

Affixing the letters with their powers, we have,

$$4a^5+0a^4b-16a^3b^2+10a^2b^3+15ab^4-25b^5=$$
$$4a^5-16a^3b^2+10a^2b^3+15ab^4-25b^5.$$

5. Multiply $5a^5-3a^2+a$ by $2a^4+a^3$.

Ans. $10a^9+5a^8-6a^6-a^5+a^4$.

6. Multiply $3x^3-2x-2$ by x^2-3.

Ans. $3x^5-11x^3-2x^2+6x+6$.

7. Multiply y^2+y-3 by y^3-y.
Ans. $y^5+y^4-4y^3-y^2+3y$.

8. Multiply $x^5+x^4+x^3+x^2+x+1$ by $x-1$.
Ans. x^6-1.

9. Multiply $a^2-2ab+4b^2$ by $a^2+2ab+4b^2$.
Ans. $a^4+4a^2b^2+16b^4$.

10. Multiply $3a^4+3a^3b+3a^2b^2+3ab^3+3b^4$ by $7a-7b$.
Ans. $21a^5-21b^5$.

11. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Ans.* x^4-y^4.

SECTION V.

DIVISION.

Art. 88. Division is the converse of Multiplication, and is performed like that of numbers. Its object is to find how many times one quantity is contained in another; or to find what quantity, multiplied by a given quantity, will produce another given quantity.

The product of like signs, as in the rule of Multiplication, produces +, and unlike signs −.

Case I.

89. When the divisor and dividend are both simple quantities.

If abc be divided by a, the quotient will be bc; because a multiplied by bc will produce abc.

If $4abc$ be divided by $2a$, the quotient will be $2bc$; because $2a$ multiplied by $2bc$ will produce $4abc$.

If $9bx$ be divided by $3x$, the quotient is $3b$; for $3b$ multiplied by $3x$ is $9bx$.

From the above illustration we derive the following

Rule. *Write the dividend over the divisor, in the manner of* *a fraction, and reduce it to its simplest form by cancelling the letters and figures that are common to all the terms.*

Or, divide the coefficient of the dividend by the coefficient of the divisor, and cancel the letters common to the divisor and dividend.

EXAMPLES.

1. Divide $6ab$ by $2a$.

$$\frac{6ab}{2a}=3b\text{; or, }6ab\div 2a=3b.$$

2. Divide $12abcxy$ by $4bx$.

$$\frac{12abcxy}{4bx}=3acy\text{; or, }12abcxy\div 4bx=3acy.$$

3. Divide $mnop$ by op. *Ans.* mn.
4. Divide $7abm$ by am. *Ans.* $7b$.
5. Divide $14xyz$ by $7x$. *Ans.* $2yz$.
6. Divide $10abcd$ by $5bcd$.
7. Divide $9mnx$ by $3x$.
8. Divide $17ab$ by ab.
9. Divide $49qrst$ by $7qt$.
10. Divide $20hmno$ by $4no$.

90. Powers and roots of the same quantity are divided by subtracting the index of the divisor from that of the dividend. Thus, if we wish to divide a^5 by a^3, we subtract the index 3 from the index 5, and set the remainder 2 over the a; thus, a^2. This process is evident from the fact that $a^5=aaaaa$, and $a^3=aaa$, and $aaaaa$ divided by aaa gives $aa=a^2$.

11. Divide $4a^3b^4$ by $2ab^2$.

$$\frac{4a^3b^4}{2ab^2}=2a^2b^2\text{; or, }4a^3b^4\div 2ab^2=2a^2b^2.$$

12. Divide $7a^6$ by a^2. *Ans.* $7a^4$.
13. Divide $6a^4b^2cd$ by $3ab$. *Ans.* $2a^3bcd$.
14. Divide $7r^4p^7$ by r^2p^2. *Ans.* $7r^2p^5$.
15. Divide $60p^7y^3$ by $30p^4$. *Ans.* $2p^3y^3$.
16. Divide $12ax^2y^2$ by $4ax^2$. *Ans.* $3y^2$.
17. Divide $96r^4st^6u^6$ by $48st^6u^2$. *Ans.* $2r^4u^4$.

18. Divide $17a^5xy^7$ by 17. *Ans.* a^5xy^7.

19. Divide $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$. *Ans.* $a^{\frac{1}{6}}$.

CASE II.

91. When the divisor is a simple quantity, and the dividend a compound one, we adopt the following

RULE. *Divide each term of the dividend by the divisor, as in* Art. 89. *Or, we may write the divisor under the dividend, in the form of a fraction, and then cancel equal quantities when found in the divisor and in each term of the dividend.*

EXAMPLES.

1. Divide $9a^3b+6a^4c-12ab$ by $3a$.

OPERATION.

$$\begin{array}{l} 3a)\underline{9a^3b+6a^4c-12ab} \\ \quad\;\; 3a^2b+2a^3c-\;4b. \; \textit{Ans.} \end{array}$$

We find that $3a$ is a factor in each term of the dividend; we therefore write the other factors under their respective quantities.

2. Divide $8a^3bc+16a^5bc-4a^2c^2$ by $4a^2c$.
Ans. $2ab+4a^3b-c$.

3. Divide $9a^5bc-3a^2b+18a^3bc$ by $3ab$.
Ans. $3a^4c-a+6a^2c$.

4. Divide $20a^4bc+15abd^3-10a^2be$ by $5ab$.
Ans. $4a^3c+3d^3-2ae$.

5. Divide $15x^2y^3+30x^5y^7$ by x^2. *Ans.*

6. Divide $7ax^4yz^3-14xyz+21xy^2$ by $7xy$. *Ans.*

7. Divide $p^2mq+p^3m-p^4mc$ by p^2. *Ans.*

8. Divide $4txz-8t^2z+z^2$ by z. *Ans.*

9. Divide $12a^{-2}-8a^2b+16a^3x-10a^{-2}y$ by $2a^2$.
Ans. $6a^{-4}-4b+8ax-5a^{-4}y$.

CASE III.

92. When the divisor and dividend are both compound quantities.

Rule. *Write down the quantities in the same manner as in the division of numbers in Arithmetic, arranging the terms of each quantity so that the highest powers of one of the letters may stand before the next lower.*

Divide the first term of the dividend by the first term of the divisor, and set the result in the quotient, with its proper sign.

Multiply the whole divisor by the term thus found; and, having subtracted the result from the dividend, bring down as many terms to the remainder as are requisite for the next operation, which perform as before; and so proceed, as in Arithmetic, till the work is finished.

1. Divide $a^2+2ab+b^2$ by $a+b$.

$$
\begin{array}{l}
a^2+2ab+b^2\left(\dfrac{a+b \text{ divisor.}}{a+b \text{ quotient.}}\right. \\
\underline{a^2+\ ab} \\
ab+b^2 \\
ab+b^2
\end{array}
$$

2. Divide $a^3+5a^2x+5ax^2+x^3$ by $a+x$.

$$
\begin{array}{l}
a^3+5a^2x+5ax^2+x^3\left(\dfrac{a+x}{a^2+4ax+x^2.}\right. \\
\underline{a^3+a^2x} \\
4a^2x+5ax^2 \\
\underline{4a^2x+4ax^2} \\
ax^2+x^3 \\
ax^2+x^3.
\end{array}
$$

3. Divide $a^4+4a^2b^2+16b^4$ by $a^2-2ab+4b^2$.

$$
\begin{array}{l}
a^4+4a^2b^2+16b^4\left(\dfrac{a^2-2ab+4b^2}{a^2+2ab+4b^2}\right. \\
\underline{a^4-2a^3b+4a^2b^2} \\
2a^3b+16b^4 \\
\underline{2a^3b-4a^2b^2+8ab^3} \\
4a^2b^2-8ab^3+16b^4 \\
\underline{4a^2b^2-8ab^3+16b^4.}
\end{array}
$$

It may be verified that $a^2+2ab+4b^2$ is the true quotient, by multiplying it by the divisor. It should also be observed, that in *every* stage of the proceeding, the terms involving the highest powers of a have been placed first on the *left*.

4. Divide $4x^4-9a^2x^2+6a^3x-a^4$ by $2x^2-3ax+a^2$.

$$\begin{array}{l}
4x^4-9a^2x^2+6a^3x-a^4\left(\dfrac{2x^2-3ax+a^2}{2x^2+3ax-a^2}\right. \\
\underline{4x^4-6ax^3+2a^2x^2\qquad\qquad} \\
\qquad 6ax^3-11a^2x^2+6a^3x \\
\qquad \underline{6ax^3-9a^2x^2+3a^3x\qquad} \\
\qquad\qquad -2a^2x^2+3a^3x-a^4 \\
\qquad\qquad -2a^2x^2+3a^3x-a^4.
\end{array}$$

93. If the divisor be not exactly contained in the dividend, the quantity that remains after the division is finished must be placed over the divisor at the right of the quotient, in the form of a fraction.

5. Divide a^3-x^3 by $a+x$.

$$\begin{array}{l}
a^3-x^3\left(\dfrac{a+x}{a^2-ax+x^2-\dfrac{2x^3}{a+x}}\right. \\
\underline{a^3+a^2x\qquad} \\
\quad -a^2x-x^3 \\
\quad \underline{-a^2x-ax^2\qquad} \\
\qquad\qquad ax^2-x^3 \\
\qquad\qquad \underline{ax^2+x^3} \\
\qquad\qquad\quad -2x^3.
\end{array}$$

94. The operation of division may be considered as terminated when the highest power of the letter, in the first or leading term of the remainder, is less than the first term of the divisor.

The division of quantities may also be sometimes carried on *ad infinitum*, like a decimal fraction; in which case a few of the leading terms of the quotient will, generally, be sufficient to indicate the rest, without its being necessary to continue the operation.

6. Divide a by $a+x$.

$$\begin{array}{l|l}
a & a+x \\ \hline
 & 1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}, \&c. \\
\underline{a+x} & \\
-x & \\
\underline{-x-\frac{x^2}{a}} & \\
\quad +\frac{x^2}{a} & \\
\quad \underline{+\frac{x^2}{a}+\frac{x^3}{a^2}} & \\
\qquad -\frac{x^3}{a^2} & \\
\qquad \underline{-\frac{x^3}{a^2}-\frac{x^4}{a^3}.} &
\end{array}$$

7. Divide a by $a-x$.

Ans. $1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\frac{x^4}{a^4}+\frac{x^5}{a^5}$, &c.

8. Let $a^2-2ax+x^2$ be divided by $a-x$. *Ans.* $a-x$.

9. Divide $a^3-3a^2b+3ab^2-b^3$ by $a-b$.

Ans. $a^2-2ab+b^2$.

10. Divide $8a^3-4a^2b-6ab^2+3b^3$ by $2a-b$.

Ans. $4a^2-3b^2$.

11. Divide $3b^3+3ab^2-4a^2b-4a^3$ by $a+b$.

Ans. $-4a^2+3b^2$.

12. Let $2a^2x^2-5ax+2$ be divided by $2ax-1$.

Ans. $ax-2$.

13. Divide $21a^5-21b^5$ by $7a-7b$.

Ans. $3a^4+3a^3b+3a^2b^2+3ab^3+3b^4$.

14. Divide $x^4-y^4+2y^2z^2-z^4$ by $x^2+y^2-z^2$.

Ans. $x^2-y^2+z^2$.

15. Divide $1+a$ by $1-a$.

Ans. $1+2a+2a^2+2a^3+2a^4+$, &c.

16. Divide $8x^2-15y^2+23yz-2xy-8xz-6z^2$ by $2x-3y+z$.

Ans. $4x+5y-6z$.

17. Divide $6x^4-96$ by $3x-6$.

Ans. $2x^3+4x^2+8x+16$.

18. Divide $a^8+a^6b^2+a^4b^4+a^2b^6+b^8$ by $a^4+a^3b+a^2b^2+ab^3+b^4$.

Ans. $a^4-a^3b+a^2b^2-ab^3+b^4$.

DIVISION BY DETACHED COEFFICIENTS.

95. As the pupil has seen in Art. 87 that the operation of many questions in Multiplication is facilitated by using detached coefficients, he will readily perceive that the same principle will apply to Division.

The terms of the divisor and dividend are to be arranged according to the power of the letters, and zero must be inserted in the terms that are wanting.

The first literal term of the quotient is obtained by dividing the first letter of the dividend by the first letter of the divisor; and the letters belonging to the other terms are written in the same order, as they are found in the divisor and dividend.

EXAMPLES.

1. Divide $a^3+3a^2b+3ab^2+b^3$ by $a+b$.

$$\begin{array}{l}
1+3+3+1\left(\dfrac{1+1}{1+2+1}\right. \begin{array}{l}\text{coefficients of the divisor.}\\ \text{coefficients of the quotient.}\end{array}\\
\underline{1+1}\\
\quad 2+3\\
\quad \underline{2+2}\\
\qquad 1+1\\
\qquad \underline{1+1.}
\end{array}$$

$a^3\div a=a^2$, first literal term of the quotient. The others will therefore be $ab+b^2$, and these terms annexed to the coefficients will be $a^2+2ab+b^2$.

2. Divide x^4-y^4 by x^2-y^2.

$$\begin{array}{l}
1+0+0+0-1\left(\dfrac{1+0-1}{1+0+1}\right.\\
1+0-1\\
\hline
\qquad\quad 1+0-1\\
\qquad\quad 1+0-1.\\
\hline
\end{array}$$

$x^4\div x^2=x^2$, first literal part. The other regular parts are $xy+y^2$. Having prefixed the coefficients, it will be $x^2+0xy+y^2$; but, as the coefficient of the second term is zero, the term has no value. The correct answer will therefore be x^2+y^2.

3. Divide $3x^4-48$ by $3x-6$.

$$\begin{array}{l}
3+0+0+0-48\left(\dfrac{3-6}{1+2+4+8}\right.\\
3-6\\
\hline
\qquad 6\\
\qquad 6-12\\
\hline
\qquad\quad 12\\
\qquad\quad 12-24\\
\hline
\qquad\qquad 24-48\\
\qquad\qquad 24-48.\\
\hline
\end{array}$$

$x^4\div x=x^3$, first literal part. The succeeding terms will, therefore, be x^2+x+x^0. Hence the true quotient will be, x^3+2x^2+4x+8.

4. Divide $1-a^8$ by $1+a$.

Ans. $1-a+a^2-a^3+a^4-a^5+a^6-a^7$.

5. Divide $3y^3+3xy^2-4x^2y-4x^3$ by $x+y$. *Ans.* $3y^2-4x^2$.

6. Divide $a^4-3a^3b-8a^2b^2+18ab^3-8b^4$ by $a^2+2ab-2b^2$.

Ans. $a^2-5ab+4b^2$.

7. Divide $m^5-5m^4n+10m^3n^2-10m^2n^3+5mn^4-n^5$ by $m^2-2mn+n^2$. *Ans.* $m^3-3m^2n+3mn^2-n^3$.

8. Divide $a^{m+n}+a^{n+1}b^{m-1}-a^{m-1}b^{n+1}-b^{m+n}$ by $a^{m-1}+b^{m-1}$.

Ans. $a^{n+1}-b^{n+1}$.

QUESTIONS TO EXERCISE THE FOREGOING RULES.

1. What is the sum of the following quantities: $12a+5c+17d+13b$, $8a+12b+15d+8c$, $11c+15a+23b+10d$, and $4d+3a+20b+18c$? *Ans.* $38a+68b+42c+46d$.

2. Add together $5a+3b-4c$, $2a-5b+6c+2d$, $a-4b-2c+3e$, and $7a+4b-3c-6e$. *Ans.* $15a-2b-3c+2d-3e$.

3. Find the sum of $3a^2+2ab+4b^2$, $5a^2-8ab+6b^2$, $-4a^2+5ab-b^2$, $18a^2-20ab-19b^2$, $14a^2-3ab+20b^2$, and $-36a^2+24ab-10b^2$. *Ans.*

4. Required the sum of $5a^2b-17a^3bc-15b^2c^4+5$, $-4a^2b+8a^3bc-10b^2c^4-4$, $-3a^2b-3a^3bc+20b^2c^4-3$, and $2a^2b+12a^3bc+5b^2c^4+2$. *Ans.*

5. Add the following quantities: $a+b+c+d$, $a+b+c-2d$, $a+b-2c+d$, $a-3b+c+d$, $-a+b+c+d$, and $a-b-2c-2d$. *Ans.* $4a$.

6. Multiply x^2+2x+1 by x^2-2x+3. *Ans.* x^4+4x+3.

7. Multiply $1-x+x^2-x^3$ by $1+x$. *Ans.* $1-x^4$.

8. Multiply $1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7$ by $1+2x+x^2$. *Ans.* $1-9x^8-8x^9$.

9. What is the continued product of $a+b$, $a-b$, a^2+ab+b^2, and a^2-ab+b^2? *Ans.* a^6-b^6.

10. Multiply $x^3+3ax^2+3a^2x+a^3$ by $x^3-3ax^2+3a^2x-a^3$. *Ans.* $x^6-3a^2x^4+3a^4x^2-a^6$.

11. Multiply $a^{m-1}+b^{m-1}$ by $a^{n+1}-b^{n+1}$. *Ans.* $a^{m+n}+a^{n+1}b^{m-1}-a^{m-1}b^{n+1}-b^{m+n}$.

12. Divide x^5-a^5 by $x-a$. *Ans.* $x^4+ax^3+a^2x^2+a^3x+a^4$.

13. Divide $x^4-9x^2-6xy-y^2$ by x^2+3x+y. *Ans.* x^2-3x-y.

14. Divide $x^4-4x^3+6x^2-4x+1$ by x^2-2x+1, and $x^4-2a^2x^2+16a^3x-15a^4$ by $x^2+2ax-3a^2$, and find the difference of their quotients. *Ans.* $2x-2ax-1+5a^2$.

15. Divide $x^6-16a^3x^3+64a^6$ by $x-2a$. *Ans.* $x^5+2ax^4+4a^2x^3-8a^3x^2-16a^4x-32a^5$.

SECTION VI.

FRACTIONS.

Art. 96. Algebraic Fractions are similar to vulgar fractions in Arithmetic; they express a part, or parts, of a quantity or a unit.

97. They consist of two parts, the *numerator* and *denominator*, the former being written *above* the line, and the latter *below* it; and these, when taken together, are the *terms* of the fraction.

98. The denominator shows into how many parts the quantity or unit is divided; and the numerator, how many of these parts are represented by the fraction.

99. A *proper fraction* is one whose numerator is less than its denominator; as,

$$\frac{a-b}{a+d}, \text{ or } \frac{7}{8}.$$

100. An *improper fraction* is one whose numerator is equal to or greater than its denominator; as,

$$\frac{a}{a}, \text{ or } \frac{b+c}{b-c}, \text{ or } \frac{7}{3}.$$

101. A *mixed quantity* is a whole number or quantity, with a fraction annexed, with the sign either plus or minus; as,

$$\frac{a}{b}+y, \text{ or } \frac{m}{n}-x, \text{ or } y+\frac{a}{c}, \text{ or } x-\frac{m}{n}, \text{ or } 7\tfrac{3}{8}.$$

102. A *compound fraction* is a fraction of a fraction; as,

$$\frac{a}{b} \text{ of } \frac{c}{d} \text{ of } \frac{m}{n}; \text{ or, } \tfrac{7}{8} \text{ of } \tfrac{5}{8} \text{ of } \tfrac{3}{10}.$$

103. A *complex fraction* is a fraction having a fraction in its numerator or denominator, or in both: as,

$$\frac{\frac{3}{4}}{14\frac{1}{2}}, \quad \frac{\frac{4}{7}}{\frac{11}{12}}, \quad \frac{\frac{a}{b}}{\frac{c}{d}}, \text{ or } \frac{\frac{a}{c-d}}{\frac{n}{p}}.$$

104. The value of a fraction depends on the ratio which the numerator bears to the denominator.

105. The value of a fraction is not changed by multiplying or dividing both numerator and denominator by the same quantity.

106. The *greatest common measure* of two or more quantities is the largest quantity that will divide all of them without a remainder.

107. The *least common multiple* of two or more quantities is the least quantity that can be divided by them all without a remainder.

108. A fraction is in its lowest terms when no quantity, excepting a unit, will divide both of its terms.

109. Quantities are said to be *prime* to one another when their *greatest common measure* is a unit.

110. *Prime factors* of quantities are those factors which can be divided by no quantity but themselves or a unit; thus, the prime factors of 35 are 7 and 5.

111. A *composite quantity* is that produced by multiplying two or more quantities together.

112. A fraction is, in value, equal to the number of times the numerator contains the denominator.

113. A fraction is increased in value either by multiplying its numerator or dividing its denominator.

114. A fraction is diminished in value either by dividing its numerator or multiplying its denominator.

CASE I.

115. To find the greatest common measure or divisor of the terms of a fraction.

RULE. *Arrange the two quantities according to the order of their powers, and divide that which is of the highest dimensions by the other, having first cancelled any factor that may be contained in all the terms of the divisor, without being common to those of the dividend.*

Divide this divisor by the remainder, simplified as before, and so on for each successive remainder, and its preceding divisor, till nothing remains; and the last divisor will be the greatest common measure or divisor required.

If any of the divisors, in the course of the operation, become negative, they may have their signs changed, or be taken affirmatively, without altering the truth of the result; and, if the first term of a divisor should not be exactly contained in the first term of the dividend, the several terms of the latter may be multiplied by any number or quantity that will render the division complete.

EXAMPLES.

1. Find the greatest common measure or divisor of $\frac{cx+x^2}{a^2c+a^2x}$.

$$\begin{array}{l} cx+x^2)a^2c+a^2x \\ \quad c+x)\,a^2c+a^2x(a^2 \\ \qquad\quad\; a^2c+a^2x. \end{array}$$

As x is found in both terms of the divisor, we divide those terms by x before the operation.

The greatest common measure of both terms we perceive is $c+x$; that is, it will divide them both without a remainder.

Thus, $c+x)\frac{cx+x^2}{a^2c+a^2x}=\frac{x}{a^2}$.

2. Required the greatest factor of $\frac{x^3-b^2x}{x^2+2bx+b^2}$.

$$\begin{array}{l}
x^2+2bx+b^2)x^3-\ b^2x(x \\
\qquad\qquad x^3+2bx^2+b^2x \\
\hline
\qquad\qquad -2bx^2-2b^2x) \\
\qquad\qquad x+b)x^2+2bx+b^2(x+b. \\
\qquad\qquad\qquad\quad x^2+\ bx \\
\hline
\qquad\qquad\qquad\qquad bx+b^2 \\
\qquad\qquad\qquad\qquad bx+b^2
\end{array}$$

We cancel $2bx$ in both terms of the second divisor, as it is common to both.

As $x+b$ is the last divisor, it is the greatest factor or common measure of the quantities proposed.

3. Required the greatest common divisor of $3a^2-2a-1$, and $4a^3-2a^2-3a+1$.

$$\begin{array}{l}
3a^2-2a-1)4a^3-2a^2-3a+1(4a \\
\qquad\qquad 3 \\
\hline
\qquad\qquad 12a^3-6a^2-9a+3 \\
\qquad\qquad 12a^3-8a^2-4a \\
\hline
\qquad\qquad\qquad 2a^2-5a+3)3a^2-2a-1 \\
\qquad\qquad\qquad\qquad\qquad\quad 2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\quad 6a^2-\ 4a-2(3 \\
\qquad\qquad\qquad\qquad\qquad\quad 6a^2-15a+9 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad 11a-11 \\
\qquad\qquad\qquad\qquad\qquad\qquad a-1)2a^2-5a+3(2a-3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2a^2-2a \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -3a+3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -3a+3. \\
\hline
\end{array}$$

As 11 is common to both terms of the third divisor, it is cancelled; therefore $a-1$ is the greatest common factor of both quantities.

4. What is the greatest common divisor of x^3-a^3, and x^2-a^2? *Ans.* $x-a$.

5. What is the greatest common factor of x^2-1, and $ax+a$? *Ans.* $x+1$.

6. Required the greatest common factor of y^4-x^4, and $y^3-y^2x-yx^2+x^3$. *Ans.* y^2-x^2.

7. Required the greatest common measure of $a^3-a^2x+ax^2-x^3$, and a^4-x^4. *Ans.* $a^3-a^2x+ax^2-x^3$.

8. Required the greatest common factor of a^4-x^4, and $a^5+a^3x^2$. *Ans.* a^2+x^2.

CASE II.

116. To reduce fractions to their lowest terms.

RULE. *Divide the terms of the fraction by the prime factors common to both.*

Or, divide both terms of the fraction by their greatest common divisor.

117. That fractions after reduction have the same value as before, is evident from the fact that their numerators retain the same ratio to their denominators; for equi-multiples and sub-multiples of any two numbers have the same ratio to each other as the numbers themselves.

Letters or numbers common to all the quantities in each term of the fraction may be cancelled.

EXAMPLES.

1. Reduce $\frac{4abc}{6a^2bd}$ to its lowest terms.

$$\frac{4abc}{6a^2bd}=\frac{2ab\times 2c}{2ab\times 3ad}=\frac{2c}{3ad}. \quad \textit{Ans.}$$

In this operation we find $2ab$ to be the largest factor in both terms; it, therefore, may be cancelled, and the answer is $\frac{2c}{3ad}$.

2. Reduce $\frac{abxy}{admny}$ to its lowest terms.

$$\frac{abxy}{admny}=\frac{bx}{dmn}. \quad \textit{Ans.}$$

In this question we find a and y common to both terms; and, they being cancelled, the result is $\frac{bx}{dmm}$.

3. Reduce $\frac{mnopq^2}{mnop^2qx}$ to its lowest terms. *Ans.* $\frac{q}{px}$.

4. Reduce $\frac{a^2bc}{5a^2b^2}$ to its lowest terms. *Ans.* $\frac{c}{5b}$.

5. Reduce $\frac{12am}{15bcm}$ to its lowest terms. *Ans.* $\frac{4a}{5bc}$.

6. Reduce $\frac{4ax^2}{36a^2x^3}$ to its lowest terms. *Ans.* $\frac{1}{9ax}$.

7. Reduce $\frac{6an^2}{36bn}$ to its lowest terms. *Ans.* $\frac{an}{6b}$.

8. Reduce $\frac{56bm^2x^6}{76bmx^2}$ to its lowest terms. *Ans.* $\frac{14mx^4}{19}$.

9. Reduce $\frac{19ab^2cde^3}{76a^3bcde^5}$ to its lowest terms. *Ans.* $\frac{b}{4a^2e^2}$.

10. Reduce $\frac{x^3-b^2x}{x^2+2bx+b^2}$ to its lowest terms.

In performing this question, we first find the greatest common measure of the two terms of the fraction, which is $x+b$; we then divide both terms by it. Thus,

$$x+b\Big)\frac{x^3-b^2x}{x^2+2bx+b^2}=\frac{x^2-bx}{x+b}. \quad \textit{Ans.}$$

11. Reduce $\frac{6a^2+5ax-6x^2}{6a^2+13ax+6x^2}$ to its lowest terms.

Ans. $\frac{3a-2x}{3a+2x}$.

12. Reduce $\frac{a^2-x^2}{a^4-x^4}$ to its lowest terms. *Ans.* $\frac{1}{a^2+x^2}$.

13. It is required to reduce $\frac{x^6-y^6}{x^4-y^4}$ to its lowest terms.

Ans. $\frac{x^4+x^2y^2+y^4}{x^2+y^2}$.

CASE III.

118. To reduce a mixed quantity to the form of a fraction.

RULE. *Multiply the integral part by the denominator of the fractional part; to this product annex the numerator of the fraction, prefixing to it the sign of the fraction; under the whole write the denominator of the fraction.*

EXAMPLES.

1. Reduce $7\frac{3}{5}$ to a fractional form. $\frac{7\times5+3}{5}=\frac{38}{5}$. *Ans.*

2. Reduce $a+\frac{b}{e}$ to the form of a fraction.

$\frac{a\times e+b}{e}=\frac{ae+b}{e}$. *Ans.*

3. Change $a+\frac{a^2-b^2}{m}$ to a fraction.

$\frac{a\times m+\overline{a^2-b^2}}{m}=\frac{am+a^2-b^2}{m}$. *Ans.*

4. Change $a-\frac{m+n}{e}$ to the form of a fraction.

$\frac{a\times e-\overline{m+n}}{e}=\frac{ae-m-n}{e}$. *Ans.*

5. Reduce $x-\frac{a-b}{m}$ to the form of a fraction.

$\frac{x\times m-\overline{a-b}}{m}=\frac{mx-a+b}{m}$. *Ans.*

6. Change $a+\frac{b^2-cd}{n}$ to the form of a fraction.

Ans. $\frac{an+b^2-cd}{n}$

7. Reduce $7x-\frac{4n^2+5a}{8}$ to the form of a fraction.

Ans. $\frac{56x-4n^2-5a}{8}$.

8. Reduce $15a-\frac{3m^2x-d^2}{4m}$ to the form of a fraction.

Ans. $\frac{60am-3m^2x+d^2}{4m}$.

9. Reduce $7a-b-\frac{7e-m}{4n}$ to the form of a fraction.

Ans. $\frac{28an-4bn-7e+m}{4n}$.

10. Change $11m-4n+\frac{a^2-5n^3}{3m-2n^2}$ to the form of a fraction.

Ans. $\frac{33m^2-22mn^2-12mn+3n^3+a^2}{3m-2n^2}$.

11. Reduce $8x^2+5y^2-\frac{d^3+a^2}{2x-3y^2}$ to the form of a fraction.

Ans. $\frac{16x^3+10xy^2-24x^2y^2-15y^4-d^3-a^2}{2x-3y^2}$.

Case IV.

119. To represent a fraction in the form of a whole or mixed quantity.

Rule. *Divide the numerator by the denominator for the integral part, and write the remainder, if any, over the denominator for the fractional part; annex this to the integral part, and it will represent the quantity required.*

EXAMPLES.

1. Change $\frac{27}{8}$ to a mixed quantity.

$\frac{27}{8}=27\div8=3\frac{3}{8}$. *Ans.*

2. Change $\frac{88}{11}$ to a whole number.

$\frac{88}{11}=88\div11=8$. *Ans.*

3. Change $\frac{ax+a^2}{x}$ to a mixed quantity.

$\frac{ax+a^2}{x}=\overline{ax+a^2}\div x=a+\frac{a^2}{x}$. *Ans.*

4. Change $\frac{ab-a^2}{b}$ to a mixed quantity.

$$\frac{ab-a^2}{b}=\overline{ab-a^2}\div b=a-\frac{a^2}{b}. \quad \textit{Ans.}$$

5. Change $\frac{a^3-b^3+x^3}{a+x}$ to its equivalent mixed quantity.

Ans. $a^2-ax+x^2-\frac{b^3}{a+x}$.

6. Change $\frac{x^3+y^3}{x+y}$ to a whole number. *Ans.* x^2-xy+y^2.

7. Change $\frac{x^3-y^3}{x-y}$ to a whole number. *Ans.* x^2+xy+y^2.

8. Find a mixed quantity equivalent to $\frac{ax^2-x}{a}$.

Ans. $x^2-\frac{x}{a}$.

CASE V.

120. To reduce a complex fraction to a simple one.

RULE. *If the numerator or denominator, or both, be whole or mixed quantities, reduce them to improper fractions. Then multiply the denominator of the lower fraction into the numerator of the upper for a new numerator and the denominator of the upper fraction into the numerator of the lower for a new denominator; or, invert the denominator of the complex fraction when reduced, and place it in a line with the numerator; then multiply the two numerators together for a new numerator, and the two denominators together for a new denominator.*

All fractions in this proposition must be reduced to this form, $\frac{\frac{a}{c}}{\frac{d}{b}}$, or $\frac{\frac{3}{4}}{\frac{2}{5}}$, before they can be solved by the above rule. Now, every fraction denotes a division of the numerator by the denominator, and its value is equal to the quotient obtained by such a division. Hence, by the nature of division, we have,

$$\frac{\frac{a}{c}}{\frac{d}{b}}=\frac{a}{c}\times\frac{b}{d}=\frac{ab}{cd}.$$

By the preceding rules we are enabled to show all the variations that can possibly happen in preparing fractions, and also the method of reducing them to their lowest terms.

EXAMPLES.

1. Reduce $\frac{\frac{7}{8}}{\frac{5}{3}}$ to a simple fraction. $\frac{\frac{7}{8}}{\frac{5}{3}}=\frac{7}{8}\times\frac{3}{5}=\frac{21}{40}$. *Ans.*

2. Reduce $\frac{7\frac{1}{4}}{8\frac{1}{2}}$ to a simple fraction.

$$\frac{7\frac{1}{4}}{8\frac{1}{2}}=\frac{\frac{29}{4}}{\frac{17}{2}}=\frac{29}{4}\times\frac{2}{17}=\frac{58}{68}=\frac{29}{34}.\quad \textit{Ans.}$$

3. Reduce $\frac{7}{\frac{1}{3}}$ to a simple fraction.

$$\frac{7}{\frac{1}{3}}=\frac{\frac{7}{1}}{\frac{1}{3}}\ \frac{7}{1}\times\frac{3}{1}=\frac{21}{1}=21.\quad \textit{Ans.}$$

4. Reduce $\frac{\frac{3}{5}}{6\frac{3}{4}}$ to a simple fraction.

$$\frac{\frac{3}{5}}{6\frac{3}{4}}=\frac{\frac{3}{5}}{\frac{27}{4}}=\frac{3}{5}\times\frac{4}{27}=\frac{12}{135}=\frac{4}{45}.\quad \textit{Ans.}$$

5. Reduce $\frac{\frac{a}{b}}{m+n}$ to a simple fraction.

$$\frac{\frac{a}{b}}{m+n}=\frac{\frac{a}{b}}{\frac{m+n}{1}}=\frac{a}{b}\times\frac{1}{m+n}=\frac{a}{bm+bn}.\quad \textit{Ans.}$$

6. Reduce $\frac{a}{x+\frac{a}{y}}$ to a simple fraction.

$$\frac{a}{x+\frac{a}{y}}=\frac{\frac{a}{1}}{\frac{xy+a}{y}}=\frac{a}{1}\times\frac{y}{xy+a}=\frac{ay}{xy+a}.\quad \textit{Ans.}$$

7. Reduce $\dfrac{a+\dfrac{b}{c}}{x-\dfrac{m}{n}}$ to a simple fraction.

$$\frac{a+\dfrac{b}{c}}{x-\dfrac{m}{n}}=\frac{\dfrac{ac+b}{c}}{\dfrac{nx-m}{n}}=\frac{ac+b}{c}\times\frac{n}{nx-m}=\frac{acn+bn}{cnx-cm}. \quad \textit{Ans.}$$

8. Reduce $\dfrac{\frac{5}{6}}{7\frac{1}{4}}$ to a simple fraction. *Ans.* $\frac{10}{87}$.

9. Reduce $\dfrac{a-\dfrac{x}{2}}{b+\dfrac{2y}{3}}$ to a simple fraction. *Ans.* $\dfrac{6a-3x}{6b+4y}$.

10. Reduce $\dfrac{m-\dfrac{n}{3}}{x}$ to a simple fraction. *Ans.* $\dfrac{3m-n}{3x}$

11. Reduce $\dfrac{y-x+\dfrac{a}{2}}{7\frac{3}{4}}$ to a simple fraction. *Ans.* $\dfrac{8y-8x+4a}{62}$.

CASE VI.

121. To reduce fractions to a common denominator.

RULE. *Multiply each numerator into all the denominators except its own for a new numerator, and all the denominators together for a common denominator.*

Or, find the least common multiple of all the denominators, and it will be the denominator required. Divide the common multiple by each of the denominators, and multiply the quotients by the respective numerators of the fractions, and their products will be the numerators required.

FIRST METHOD.

1. Reduce $\frac{5}{12}$, $\frac{7}{8}$, and $\frac{3}{4}$, to a common denominator.

$5\times 8\times4=160$, numerator for $\frac{5}{12}=\frac{160}{384}$.
$7\times12\times4=336$, numerator for $\frac{7}{8}=\frac{336}{384}$.
$3\times12\times8=288$, numerator for $\frac{3}{4}=\frac{288}{384}$.
$12\times 8\times4=384$, common denominator.

Equimultiples of the terms of a fraction express the same value as the fraction itself. The terms of $\frac{5}{12}$ are each multiplied by 8 and 4. Hence $\frac{160}{384}$ has the same value as $\frac{5}{12}$. The same may be observed of $\frac{7}{8}$ and $\frac{3}{4}$.

2. Reduce $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{m}{n}$, to a common denominator.

$a\times d\times n=adn=$numerator of $\frac{a}{b}=\frac{adn}{bdn}$.

$c\times b\times n=bcn=$numerator of $\frac{c}{d}=\frac{bcn}{bdn}$.

$m\times b\times d=bdm=$numerator of $\frac{m}{n}=\frac{bdm}{bdn}$

$b\times d\times n=bdn=$common denominator.

SECOND METHOD.

3. Reduce $\frac{7}{8}$, $\frac{5}{12}$, and $\frac{1}{4}$, to a common denominator.

$$\begin{array}{r|l} 4 & 8,\ 12,\ 4 \\ \hline & 2,\ 3,\ 1 \end{array}\ ;\ 4\times2\times3=24, \text{ common denominator.}$$

$$\begin{array}{r|l} & 24 \\ \hline 8 & 3\times7=21, \text{ numerator for } \frac{7}{8}=\frac{21}{24}. \\ 12 & 2\times5=10, \text{ numerator for } \frac{5}{12}=\frac{10}{24}. \\ 4 & 6\times1=\ 6, \text{ numerator for } \frac{1}{4}=\frac{6}{24}. \end{array}$$

4. Reduce $\frac{a}{4x}$, $\frac{b}{x^2}$, and $\frac{3a}{8x}$, to a common denominator.

$$\begin{array}{r|l} x & 4x,\ x^2,\ 8x \\ \hline 4 & 4,\ x,\ 8 \\ \hline & 1,\ x,\ 2 \end{array}\ ;\ x\times4\times x\times2=8x^2, \text{ common denominator.}$$

$$\begin{array}{r|l} & 8x^2 \\ \hline 4x & 2x\times a=2ax, \text{ numerator to } \frac{a}{4x}=\frac{2ax}{8x^2}. \\ x^2 & 8\ \times b=8b, \text{ numerator to } \frac{b}{x^2}=\frac{8b}{8x^2}. \\ 8x & x\times3a=3ax, \text{ numerator to } \frac{3a}{8x}=\frac{3ax}{8x^2}. \end{array}$$

5. Reduce $\frac{4}{9}$, $\frac{7}{12}$, and $\frac{1}{2}$, to a common denominator.

Ans. $\frac{16}{36}$, $\frac{21}{36}$, $\frac{18}{36}$.

6. Reduce $\frac{7}{11}$, $\frac{4}{15}$, $\frac{5}{7}$, and 7, to a common denominator.

Ans.

7. Reduce $\frac{2}{3}$ of $7\frac{1}{4}$ and $\frac{2}{11}$ of 5 to a common denominator.

Ans.

8. Reduce $\frac{3}{4}$ of $\frac{8}{11}$ of 17 and $\frac{1}{2}$ of 19 to a common denominator.

Ans.

9. Reduce $\frac{\frac{3}{8}}{\frac{5}{9}}$ and $\frac{2}{11}$ of $\frac{\frac{3}{4}}{7\frac{1}{4}}$ to a common denominator.

Ans. $\frac{8613}{12760}$, $\frac{240}{12760}$.

10. Reduce $\frac{3x}{y}$, $\frac{4m}{x}$, and $\frac{a}{b-c}$, to a common denominator.

Ans. $\frac{3bx^2-3cx^2}{bxy-cxy}$, $\frac{4bmy-4cmy}{bxy-cxy}$, $\frac{axy}{bxy-cxy}$.

11. Reduce $\frac{a}{x}$, $\frac{b}{x-2}$, and $\frac{d-3}{y}$, to a common denominator.

Ans. $\frac{axy-2ay}{x^2y-2xy}$, $\frac{bxy}{x^2y-2xy}$, $\frac{dx^2-3x^2-2dx+6x}{x^2y-2xy}$.

12. Reduce $\frac{a+b}{x}$, $\frac{3-a}{y-2}$, and $\frac{x-7}{18}$, to a common denominator.

Ans. $\frac{18ay+18by-36a-36b}{18xy-36x}$, $\frac{54x-18ax}{18xy-36x}$, $\frac{x^2y-7xy-2x^2+14x}{18xy-36x}$.

13. Reduce $\frac{4a}{b-3}$, $\frac{a}{b}$, $\frac{d}{x}$, and $\frac{a-b}{x-5}$, to a common denominator.

Ans. $$\begin{cases} \frac{4abx^2-20abx}{b^2x^2-3bx^2-5b^2x+15bx}, & \frac{abx^2-3ax^2-5abx+15ax}{b^2x^2-3bx^2-5b^2x+15bx}, \\ \frac{b^2dx-3bdx-5b^2d+15bd}{b^2x^2-3bx^2-5b^2x+15bx}, & \frac{ab^2x-b^3x-3abx+3b^2x}{b^2x^2-3bx^2-5b^2x+15bx}. \end{cases}$$

14. Reduce x, y, $\frac{a}{x}$, and $\frac{a}{y-3}$, to a common denominator.

Ans. $\frac{x^2y-3x^2}{xy-3x}$, $\frac{xy^2-3xy}{xy-3x}$, $\frac{ay-3a}{xy-3x}$, $\frac{ax}{xy-3x}$.

15. Reduce a, b, c, d, and $\frac{a}{b}$, to a common denominator.

Ans. $\frac{ab}{b}$, $\frac{b^2}{b}$, $\frac{bc}{b}$, $\frac{bd}{b}$, $\frac{a}{b}$.

16. Change $\dfrac{\frac{x}{y}}{\frac{m}{2}}$ and $\dfrac{\frac{x}{3y}}{\frac{1}{m-n}}$ to a common denominator.

Ans. $\dfrac{6x}{3my}$ and $\dfrac{m^2x-mnx}{3my}$.

17. Change $\dfrac{x}{7\frac{1}{2}}$ and $\dfrac{5\frac{1}{3}}{x}$ to a common denominator.

Ans. $\dfrac{2x^2}{15x}$ and $\dfrac{80}{15x}$.

CASE VII.

ADDITION OF FRACTIONS.

122. To add fractional quantities.

RULE. *Reduce the fractions to a common denominator, and write the sum of the numerators over the common denominator.*

EXAMPLES.

1. Add $\frac{7}{8}$, $\frac{5}{12}$, and $\frac{11}{16}$, together.

$$\text{Here}\quad \left.\begin{array}{r} 7\times12\times16=1344 \\ 5\times\ 8\times16=\ 640 \\ 11\times\ 8\times12=1056 \end{array}\right\}\text{ the new numerators.}$$

$$\frac{3040}{}=1\tfrac{47}{48}.\quad \textit{Ans.}$$

And $8\times12\times16=1536$, the common denominator.

2. What is the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$?

$$\text{Here}\quad \left.\begin{array}{l} a\times d\times f=adf \\ c\times b\times f=cbf \\ e\times b\times d=ebd \end{array}\right\}\text{ the new numerators.}$$

And $b\times d\times f=bdf$, the common denominator.

Therefore, $\dfrac{adf}{bdf}+\dfrac{cbf}{bdf}+\dfrac{ebd}{bdf}=\dfrac{adf+cbf+ebd}{bdf}$. *Ans.*

3. Add the following quantities, $a-\frac{3x^2}{b}$ and $b+\frac{2ax}{c}$.

$$a-\frac{3x^2}{b}=\frac{ab-3x^2}{b};\ b+\frac{2ax}{c}=\frac{bc+2ax}{c}.$$

$$\left.\begin{array}{l}\overline{ab-3x^2}\times c=abc-3cx^2\\ \overline{bc+2ax}\times b=b^2c+2abx\end{array}\right\}\text{numerators.}$$

$$b\times c \qquad =bc, \text{ common denominator.}$$

$$\frac{abc-3cx^2}{bc}+\frac{b^2c+2abx}{bc}=\frac{abc-3cx^2+b^2c+2abx}{bc}=a+b+$$

$\frac{2abx-3cx^2}{bc}$. *Ans.*

4. Add together $\frac{3a}{5d}$, $\frac{4m}{7a}$, and $\frac{3e}{4n^2}$.

Ans. $\frac{84a^2n^2+80dmn^2+105ade}{140adn^2}$.

5. What is the sum of $\frac{7}{8}$, $\frac{5}{12}$, and $\frac{4}{9}$? *Ans.* $1\frac{53}{72}$.

6. What is the sum of $\frac{5}{8}$, $\frac{7}{11}$, and $\frac{4}{5}$? *Ans.* $2\frac{27}{440}$.

7. What is the sum of $\frac{8}{9}$, $\frac{2}{3}$, $\frac{5}{6}$, and $\frac{7}{12}$? *Ans.* $2\frac{35}{36}$.

8. What is the sum of $8\frac{3}{4}$, $3\frac{2}{3}$, and $7\frac{5}{6}$? *Ans.* $20\frac{1}{4}$.

9. What is the sum of $\frac{2}{3}$ of $7\frac{1}{4}$, and $\frac{7}{11}$ of 13? *Ans.* $13\frac{7}{66}$.

10. What is the sum of $\frac{2}{5}$ of 1, and $\frac{1}{2}$ of $\frac{2}{3}$? *Ans.* $\frac{11}{15}$.

11. What is the sum of $\frac{\frac{3}{4}}{7}$ and $\frac{\frac{1}{2}}{\frac{7}{11}}$? *Ans.* $\frac{25}{28}$.

12. What is the sum of $\frac{3}{4}$ of $\frac{4\frac{2}{3}}{11\frac{1}{2}}$ and $\frac{2}{3}$ of $\frac{\frac{3}{4}}{7\frac{2}{5}}$? *Ans.* $\frac{1688}{5106}$.

13. Find the sum of $\frac{3x}{4a}$ and $\frac{2x}{3e}$. *Ans.* $\frac{9ex+8ax}{12ae}$.

14. Find the sum of $\frac{x}{3}$, $\frac{x}{4}$, $\frac{x}{5}$. *Ans.* $\frac{47x}{60}$.

15. Find the sum of $\frac{4a}{7}$ and $\frac{a-3}{4}$. *Ans.* $\frac{23a-21}{28}$.

16. Find the sum of $4m$, $\frac{3a-1}{2}$, and $\frac{4n+2}{3}$.

Ans. $\frac{9a+24m+8n+1}{6}$.

17. What is the sum of $\frac{4a-3}{5}$, $\frac{7a+1}{8}$, and $\frac{8a}{2}$?

Ans. $\frac{139a-8}{30}$.

18. Add $\frac{3}{a+b}$ and $\frac{3}{a-b}$ together.

Ans. $\frac{6a}{a^2-b^2}$.

19. Add $\frac{a}{a-b}$, $\frac{a-b}{c+d}$, and $\frac{a+b}{c-d}$ together.

Ans. $\frac{2a^2c+ac^2-ad^2-2abc+2abd-2b^2d}{ac^2-ad^2+bd^2-bc^2}$.

20. Add $\dfrac{\frac{1}{3b^2}}{\frac{a-b}{2}}$ to $\dfrac{\frac{b}{c}}{\frac{2a-b}{3}}$. *Ans.* $\frac{4ac-2bc+9ab^3-9b^4}{6a^2b^2c-9ab^3c+3b^4c}$.

CASE VIII.

SUBTRACTION OF FRACTIONS.

123. To subtract one fraction from another.

Rule. *Reduce the fractions to a common denominator, subtract the numerator of the subtrahend from the numerator of the minuend, and write the difference over the common denominator.*

EXAMPLES.

1. From $\frac{7}{9}$ take $\frac{4}{11}$.

Here $\left.\begin{array}{l}7\times11=77\\ 4\times\ 9=36\end{array}\right\}$ the new numerators.

And $9\times11=99$, the common denominator.

Whence $\frac{77}{99}-\frac{36}{99}=\frac{41}{99}$. *Ans.*

2. From $\frac{a}{b}$ take $\frac{c}{d}$.

Here $\left.\begin{array}{l}a\times d=ad\\ c\times b=bc\end{array}\right\}$ the new numerators.

And $b\times d=bd$, the common denominator.

Whence $\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}$. *Ans.*

3. From $\frac{7}{8}$ take $\frac{4}{11}$. *Ans.* $\frac{45}{88}$.

4. From $\frac{2}{3}$ take $\frac{4}{9}$. *Ans.* $\frac{2}{9}$.

5. From $7\frac{3}{4}$ take $4\frac{1}{4}$. *Ans.* $3\frac{1}{2}$.

6. From $6\frac{3}{11}$ take $\frac{2}{3}$ of 5. *Ans.* $2\frac{31}{33}$.

7. From $8\frac{1}{7}$ take $\frac{2}{5}$ of $17\frac{1}{2}$. *Ans.* $1\frac{1}{7}$.

8. From $\frac{3}{7}$ of $11\frac{2}{15}$ take $\frac{11}{12}$ of $3\frac{1}{2}$. *Ans.* $1\frac{473}{840}$.

9. From $\frac{7}{8}$ of $13\frac{3}{7}$ take $\frac{4}{11}$ of $7\frac{1}{8}$. *Ans.* $9\frac{7}{44}$.

10. From $\frac{3}{5}$ of 7 take $\frac{1}{9}$ of $17\frac{1}{4}$. *Ans.* $2\frac{17}{60}$.

11. From $\dfrac{7\frac{2}{3}}{9\frac{1}{4}}$ take $\dfrac{\frac{2}{3}}{4\frac{2}{5}}$. *Ans.* $\frac{827}{1221}$.

12. Take $\dfrac{2}{a+1}$ from $\dfrac{2}{a-1}$. *Ans.* $\dfrac{4}{a^2-1}$.

13. From $\dfrac{7x}{5}$ take $\dfrac{4x}{7}$. *Ans.* $\dfrac{29x}{35}$.

14. From $\dfrac{3a-2b}{3c}$ take $\dfrac{2a-4b}{5b}$.

Ans. $\dfrac{15ab-6ac-10b^2+12bc}{15bc}$.

15. Required the difference of $\dfrac{12x}{5}$ and $\dfrac{3x}{7}$. *Ans.* $\dfrac{69x}{35}$.

16. Subtract $\dfrac{x-y}{x+y}$ from $\dfrac{x+y}{x-y}$. *Ans.* $\dfrac{4xy}{x^2-y^2}$.

17. Subtract $a-\dfrac{a-b}{d}$ from $3a+\dfrac{a+b}{d}$. *Ans.* $2a+\dfrac{2a}{d}$.

18. Subtract $x-\dfrac{4a-b}{2}$ from $7x-\dfrac{3a-2b}{3}$. *Ans.* $6x+a+\dfrac{b}{6}$.

19. From $\dfrac{(a+b)^2}{ab}$ take $\dfrac{(a-b)^2}{ab}$. *Ans.* 4.

20. From $\dfrac{\frac{a+b}{2}}{\frac{a-b}{3}}$ take $\dfrac{\frac{a-b}{3}}{\frac{a+b}{2}}$. *Ans.* $\dfrac{5a^2+26ab+5b^2}{6a^2-6b^2}$.

CASE IX.

MULTIPLICATION OF FRACTIONS.

124. To multiply fractions together.

RULE. *Multiply the numerators together for a new numerator, and the denominators for a new denominator.*

When the numerator of one of the fractions and the denominator of the other can be divided by some quantity which is common to each of them, the quotients may be used instead of the fractions themselves.

Also, when a fraction is to be multiplied by an integer, it is the same whether the numerator is multiplied by it or the denominator is divided by it.

If an integer is to be multiplied by a fraction, or a fraction by an integer, the integer may be considered as having unity for its denominator.

A mixed quantity should be reduced to an improper fraction.

Powers or roots of the same quantity are multiplied together by adding their indices.

EXAMPLES.

1. Multiply $\frac{3}{4}$ by $\frac{7}{8}$. $\frac{3}{4}\times\frac{7}{8}=\frac{21}{32}$. *Ans.*

2. Multiply $\frac{a}{b}$ by $\frac{m}{n}$. $\frac{a}{b}\times\frac{m}{n}=\frac{am}{bn}$. *Ans.*

3. Multiply $\frac{abc}{mn}$ by $\frac{mh}{bcd}$. $\frac{abc}{mn}\times\frac{mh}{bcd}=\frac{ah}{nd}$. *Ans.*

NOTE.—The b, c and m, are cancelled in both factors.

4. Multiply $\frac{3a}{7bc}$ by $2m$. $\frac{3a}{7bc}\times\frac{2m}{1}=\frac{6am}{7bc}$. *Ans.*

5. Multiply $a+\frac{hy}{a}$ by $\frac{m}{n}$.

$$a+\frac{hy}{a}=\frac{a^2+hy}{a}.\quad \frac{a^2+hy}{a}\times\frac{m}{n}=\frac{a^2m+hmy}{an}.\quad \textit{Ans.}$$

6. Multiply $\frac{m+n}{x+y}$ by $\frac{m+n}{x+y}$.

$$\frac{m+n}{x+y}\times\frac{m+n}{x+y}=\frac{m^2+2mn+n^2}{x^2+2xy+y^2}.\quad \textit{Ans.}$$

7. Multiply $\frac{a^2bc}{mn^2}$ by $\frac{a^3b^2}{m^3nd}$. *Ans.* $\frac{a^5b^3c}{m^4n^3d}$.

8. Multiply $\frac{3a^3x}{7h^4}$ by $\frac{4ab}{5hm}$. *Ans.* $\frac{12a^4bx}{35h^5m}$.

9. Multiply $\frac{5ab}{7hy}$ by $\frac{2a^3c}{3hy^3}$. *Ans.* $\frac{10a^4bc}{21h^2y^4}$.

10. Multiply $\frac{m^2n}{hy}$ by $\frac{mn^3}{ay^3}$. *Ans.* $\frac{m^3n^4}{ahy^4}$.

11. Multiply $\frac{3abx^2}{5xy^2}$ by $\frac{5xy^2}{3abx^2}$. *Ans.* 1.

When the multiplier and denominator of the fraction are the same quantity, they cancel each other.

12. Multiply $\frac{3ab}{7mn}$ by $7mn$. *Ans.* $3ab$.

13. Multiply $\frac{6mn}{11ab}$ by $11ab$. *Ans.* $6mn$.

14. Multiply $\frac{4acd}{xy}$ by xy. *Ans.* $4acd$.

15. Multiply $\frac{3hm}{17ab}$ by $17\,ab$. *Ans.* $3hm$.

16. Multiply $47ab$ by $\frac{x}{47ab}$. *Ans.* x.

17. Multiply $\frac{ab^{\frac{3}{4}}}{mn^{\frac{1}{2}}}$ by $\frac{b^{\frac{1}{2}}}{mn^{\frac{1}{3}}}$. *Ans.* $\frac{ab^{\frac{5}{4}}}{m^2n^{\frac{5}{6}}}$.

18. Multiply $\frac{mn^a}{4hy}$ by $\frac{hy}{mn^a}$. *Ans.* $\frac{1}{4}$.

19. Multiply $\frac{a^n}{b}$ by $\frac{a^m}{h}$. *Ans.* $\frac{a^{m+n}}{bh}$.

CASE X.

DIVISION OF FRACTIONS.

125. To divide one fraction by another.

Rule. *Multiply the denominator of the divisor by the numerator of the dividend for the numerator, and the numerator of the divisor by the denominator of the dividend for the denominator.*

Or, invert the divisor, and proceed as in multiplication.

Or, divide the numerators by each other, and the denominators by each other, when this can be done without a remainder.

Mixed quantities should be changed to improper fractions.

EXAMPLES.

1. Divide $\frac{a}{4}$ by $\frac{3a}{8}$. $\frac{a}{4}\div\frac{3a}{8}=\frac{a}{4}\times\frac{8}{3a}=\frac{8a}{12a}=\frac{2}{3}$. *Ans.*

2. Divide $\frac{3a}{2b}$ by $\frac{5c}{4d}$.

$$\frac{3a}{2b}\div\frac{5c}{4d}=\frac{3a}{2b}\times\frac{4d}{5c}=\frac{12ad}{10bc}=\frac{6ad}{5bc}. \quad \textit{Ans.}$$

3. Divide $\frac{2a+b}{3a-2b}$ by $\frac{3a+2b}{4a+b}$.

$$\frac{2a+b}{3a-2b}\times\frac{4a+b}{3a+2b}=\frac{8a^2+6ab+b^2}{9a^2-4b^2}. \quad \textit{Ans.}$$

4. Divide $\frac{3x}{4}$ by $\frac{11}{12}$. *Ans.* $\frac{36x}{44}=\frac{9x}{11}$.

5. Divide $\frac{6x^2}{5}$ by $3x^2$. *Ans.* $\frac{6x^2}{15x^2}=\frac{2}{5}$.

6. Divide $\frac{7m^2}{2}$ by $\frac{3n^2}{13}$. *Ans.* $\frac{91m^2}{6n^2}$.

7. Divide $\frac{11xy^2}{6}$ by 11. *Ans.* $\frac{xy^2}{6}$.

8. Divide $\frac{1}{xy}$ by xy. *Ans.* $\frac{1}{x^2y^2}$.

9. Divide $\frac{3x+1}{9}$ by $\frac{4x}{3}$. *Ans.* $\frac{3x+1}{12x}$.

10. Divide $\frac{32xy}{25}$ by $\frac{8x}{5}$. *Ans.* $\frac{4y}{5}$.

11. Divide $\frac{4x}{2x-1}$ by $\frac{2}{x+1}$. *Ans.* $\frac{2x^2+2x}{2x-1}$.

12. Divide $\frac{2a-b}{4ac}$ by $\frac{2ac}{3b-1}$. *Ans.* $\frac{-2a+b-3b^2+6ab}{8a^2c^2}$.

13. Divide $\frac{5a^4-5b^4}{2a^2-4ab+2b^2}$ by $\frac{6a^2+5ab}{4a-4b}$.

Ans. $\frac{20a^5-20ab^4-20a^4b+20b^5}{12a^4-14a^3b-8a^2b^2+10ab^3}$.

NEGATIVE EXPONENTS.

If we divide a^5 successively by a, the following will be the quotients:

$$a^4,\ a^3,\ a^2,\ a^1,\ 1,\ \frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \frac{1}{a^4},\ \&c.$$

By examining the above, we perceive that the exponent of each term is one less than the preceding; therefore the division might have been expressed thus:

$$a^4,\ a^3,\ a^2,\ a^1,\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4}.$$

By comparing the last quotients with the former, we find,

$$a^4=a^4;\ \ a^3=a^3;\ \ a^2=a^2;\ \ a=a^1;\ \ 1=a^0;\ \ \frac{1}{a}=a^{-1};\ \ \frac{1}{a^2}=a^{-2};$$

$$\frac{1}{a^3}=a^{-3};\ \frac{1}{a^4}=a^{-4}.$$

We also perceive that exponential quantities are divided by subtracting their indices.

Hence, if a^{-6} be divided by a^{-4}, the quotient will be $a^{-6-4}=a^{-2}$; or, x^{-m} by $x^{-n}=x^{-m-n}$.

We also infer from the above illustration that

$$\frac{a^5}{a^6},\ \frac{a^4}{a^6},\ \frac{a^3}{a^6}=a^{-1},\ a^{-2},\ a^{-3}=\frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3}.$$

Again, we see from the above that any quantity which has zero for its exponent is equal to 1.

We infer, also, that if similar quantities with negative exponents are divided by *subtracting* their indices, that such quantities are multiplied by *adding* their indices.

Thus, $a^{-2} \times a^{-3} = a^{-5}$, and $a^{3} \times a^{-3} = a^{3-3} = a^{0} = 1$.

EXAMPLES.

1. Divide a^{-5} by a^{-2}. *Ans.* a^{-3}.
2. Divide m^{-3} by m^{-5}. *Ans.* m^{2}.
3. Divide x^{4} by x^{-4}. *Ans.* x^{8}.
4. Divide $7x^{-2}$ by x^{-3}. *Ans.* $7x$.
5. Divide $8y^{-2}$ by y^{2}. *Ans.* $8y^{-4}$.
6. Multiply a^{-2} by $7a^{-3}$. *Ans.* $7a^{-5}$.
7. Multiply $3m$ by m^{-4}. *Ans.* $3m^{-3}$.
8. Multiply $4x^{-1}$ by x^{0}. *Ans.* $4x^{-1}$.
9. Multiply $a^{-1}\,b^{-2}\,c^{-3}$ by $a^{4}\,b^{3}\,c^{2}$. *Ans.* $a^{3}\,b\,c^{-1}$.
10. Divide $a^{\frac{3}{4}}$ by $a^{-\frac{4}{5}}$. *Ans.* $a^{\frac{31}{20}}$.
11. Multiply $n^{\frac{3}{5}}$ by $n^{-\frac{4}{5}}$. *Ans.* $n^{-\frac{1}{5}}$.

To free fractions from negative exponents.

RULE. *Transfer the letters which have negative exponents in the numerator to the denominator, and those which have negative exponents in the denominator to the numerator, and then change the sign of the exponent.*

NOTE. This rule implies the multiplying of all the terms of the numerator and denominator by the same quantity. Therefore, by Art. 121, the value of the fraction is the same.

EXAMPLES.

1. Free the fraction $\frac{a^{-2}\,b^{-3}}{d^{-2}\,e^{-1}}$ from negative exponents.

Ans. $\frac{d^{2}e}{a^{2}b^{3}}$.

2. Free the fraction $\frac{m\,n^{-3}\,p^{-2}}{x\,y^{-4}\,z^{-1}}$ from negative exponents.

Ans. $\frac{my^{4}z}{xn^{3}p^{2}}$.

3. Free the fraction $\frac{x^2+y^{-3}}{mn^{-4}e^{-2}}$ from negative exponents.

Ans. $\frac{e^2n^4x^2y^3+e^2n^4}{my^3}$.

4. Free the fraction $\frac{1-a^{-2}-y^2}{1-x^{-3}y^{-2}+x^{-2}}$ from negative exponents.

Ans. $\frac{a^2x^3y^2-x^3y^2-a^2x^3y^4}{a^2x^3y^2-a^2+a^2xy^2}$.

5. Free the fraction $\frac{x^{-3}y^{-2}z^{-1}}{1}$ from negative exponents.

Ans. $\frac{1}{x^3y^2z}$.

6. Free the fraction $\frac{7a^{-3}}{a^{-1}b^{-3}c^{-4}d^{-7}e^3}$ from negative exponents.

Ans. $\frac{7b^3c^4d^7}{a^2e^3}$.

SECTION VII.

EQUATIONS.

Art. 126. The doctrine of equations is that branch of Algebra which treats of the method of determining the values of unknown quantities by means of their relations to others that are known.

This is effected by making certain algebraic expressions equal to each other; which formula, in that case, is called an *equation.*

127. The *terms* of an equation are the quantities of which it is composed; and the parts that stand on each side of the sign $=$ are called the two members, or sides, of the equation.

Thus, if $x=a+b$, the terms are x, a, and b; and the meaning of the expression is, that some quantity x, standing on the left side of the equation, is equal to the sum of the quantities a and b, on the right side.

128. A *simple equation* is one which contains only the first power of the unknown quantity; as,

$$x+a=10;\ ax+bx=c;\ \text{or},\ 4x+\frac{x}{4}=17;$$

in which equation x denotes the unknown quantity, and the other letters and the numbers the known quantities.

129. A *compound equation* is one which contains two or more different powers of the unknown quantity; as, $x^2+ax=d$; or, $x^3-4x^2+3x=30$.

130. A *quadratic equation* is one in which the highest power of the unknown quantity is a square.

131. A *cubic equation* is one in which the highest power of the unknown quantity is a cube; as,

$$x^3=64;\ \text{or},\ x^3-ax^2+bx=c.$$

132. The *root* of an equation is such a quantity as, being substituted for the unknown quantity, will make both sides of the equation vanish, or become equal to each other.

133. A *simple equation* can have but one root; but every *compound equation* has as many roots as it has powers.

134. *Identical equations* are those which have the terms of the equation the same.

135. *Numerical equations* are those which contain numbers only in connection with the unknown quantities; as,

$$x^2+7x+5=100.$$

136. *Literal equations* are those in which numbers are represented by letters; thus,

$$x^2+px+ap=r.$$

137. To reduce an equation is to discover the value of the unknown quantity in it.

138. The process of reducing equations depends upon the following simple principles or axioms;

1. **If to equal quantities we add the same, or equal quantities, the sums will be equal.**

2. If from equal quantities we subtract the same, or equal quantities, the remainders will be equal.

3. If we multiply equal quantities by the same quantity, the products will be equal.

4. If we divide equal quantities by the same quantity, the quotients will be equal.

5. If we extract the same roots of equal quantities, those roots will be equal.

6. If we raise equal quantities to the same powers, those powers will be equal.

139. The known and unknown terms of an equation may be combined in various ways.

1. By addition; as, $x+7=16$, or $x+a=b$.

2. By subtraction; as, $x-9=19$, or $x-a=b$.

3. By multiplication; as, $7x=84$, or $ax=c$.

4. By division; as, $\frac{x}{4}=12$, or $\frac{x}{a}=d$.

5. By a combination of two or more of these rules; as, $\frac{3x}{4}+17=3x-5$; or, $\frac{ax}{b}+m=cx-n$.

I.

140. To find the value of the unknown quantity, when combined with a known quantity, by addition or subtraction.

1. Let $x+7=16$; and it is required to find the value of x.

Now, as $x+7$ is equal to 16, it is evident, from the second axiom, that, if from each of these equal quantities we subtract the same quantity, the two remainders will be equal. We therefore subtract 7 from each member of the equation.

Thus, $x+7-7=16-7$.

As the plus 7 and minus 7 in the first member of the equation cancel each other, the equation will be

$$x=16-7=9.$$

Therefore the value of x is 9; but, in the operation, we have only transposed the plus 7 from the first member of the equation to the second, and changed it to a minus.

2. Again, let $x-5=12$; it is required to find the value of x.

Now, by the first axiom, we find, if equals be added to equals, their sums will be equal; we therefore add 5 to each member of the equation, and we have

$$x-5+5=12+5.$$

In the first member of the equation, we have -5 and $+5$; and, as they will cancel each other, the equation will stand

$$x=12+5=17.$$

Therefore, the value of x is 17.

All that we virtually have done in the above operation has been to transpose the minus 5, in the first member of the equation, to the second, and to change it to plus.

From the foregoing examples and illustrations, we deduce the following

Rule. *When a quantity is transposed from one member of the equation to the other, the signs must be changed.*

3. Given $x+15-5=86-8$ to find the value of x.

By transposing, $x=86-8-15+5$.
By uniting, $x=68$.

4. Given $x-29+3=100-19+3$ to find the value of x.

By transposing, $x=100-19+3+29-3$.
By uniting, $x=110$.

5. Given $x+12-3=7-4$ to find the value of x.

By transposing, $x=7-4-12+3$.
By uniting, $x=-6$.

6. Given $x-5-4=24+7$ to find the value of x.

Ans. $x=40$.

II.

141. When the known and unknown quantities are combined by multiplication.

7. What is the value of x in the equation $5x+18=58$?

By transposition, $5x=58-18$.
By reduction, $5x=40$.
By division, $x=8$.

We say that if 5 times x is equal to 40, it is evident that $\frac{1}{5}$ of 5 times x, that is, x, is equal to 8.

142. Hence, if the unknown quantity in any equation be multiplied by any number or quantity, in order to find its value, *we divide the sum of all the quantities, after being reduced, by the coefficient of the unknown quantity.*

8. What is the value of x in the following equation,

$$7x-28=46+10?$$

By transposition, $7x=46+10+28$.
By reduction, $7x=84$.
By division, $x=12$.

9. Given $4x-5=71+8$ to find x. *Ans.* 21.
10. Given $6x-17-7=0$ to find x. *Ans.* 4.
11. Given $5x+28+8=6$ to find x. *Ans.* -6.
12. Given $7x-17+3=100$ to find x. *Ans.* $16\frac{2}{7}$.
13. Given $23x-96+1=0$ to find x. *Ans.* $4\frac{3}{23}$.
14. Given $17x-7-5-8=4$ to find x. *Ans.* $1\frac{7}{17}$.
15. Given $9x=7+8+10$ to find x. *Ans.* $2\frac{7}{9}$.
16. Given $7x-10=5x+14$ to find x. *Ans.* 12.

III.

143. To reduce an equation when the known and unknown quantities are combined by division.

17. Given $\frac{x}{4}=8$ to find the value of x.

Multiplying both terms by 4, we have $x=32$.

Therefore, if both terms of an equation be multiplied by any number, their products, by axiom third, are equal.

144. If a fraction be multiplied by its denominator, the product is the numerator, and the denominator disappears.

18. Given $\frac{3x}{5}=9$ to find the value of x.

Multiplying by 5, $3x=45.$
Dividing by 3, $x=15$

19. Given $\frac{ax}{d}=c$ to find the value of x.

Multiplying by d, $ax=cd.$
Dividing by a, $x=\frac{cd}{a}.$

20. Given $\frac{x}{2}+\frac{2x}{3}-\frac{3x}{5}=17$ to find the value of x.

Multiplying by 2, $x+\frac{4x}{3}-\frac{6x}{5}=34.$
Multiplying by 3, $3x+4x-\frac{18x}{5}=102.$
Multiplying by 5, $15x+20x-18x=510.$
Uniting the terms, $17x=510.$
Dividing by 17, $x=30.$

145. Hence an equation may be cleared of fractions by multiplying each term of the equation by the several denominators.

21. Given $\frac{3x}{4}+\frac{5x}{6}-\frac{3x}{8}-\frac{x}{12}=9$ to find the value of x.

The least common multiple of the denominators 4, 6, 8, and 12, is 24; and, multiplying each member of the equation by this number, we obtain

$$18x+20x-9x-2x=216.$$

Uniting the terms, $27x=216.$
Dividing by 27, $x=8.$

146. Hence an equation may be cleared of fractions by multiplying each term of the equation by the least common multiple of the denominators.

22. A boy being asked how many cents he had, replied, that if he had $\frac{3}{4}$ and $\frac{5}{6}$ as many, in addition to what he now had, he should have 62. Required the number he had.

Let x represent the number.

Then, $$\frac{3x}{4}+\frac{5x}{6}+x=62.$$

By multiplying all the terms of the equation by the least common multiple of the denominators, 4 and 6, which is 12, we have

$$9x+10x+12x=744.$$

Collecting the x's, $31x=744.$

Dividing by 31, $x=24.$ *Ans.*

VERIFICATION.

$$\frac{3\times24}{4}+\frac{5\times24}{6}+24=62.$$

$$18+20+24=62.$$

23. Given $\frac{15-x}{4}+3=6$ to find x.

Multiplying by 4, $15-x+12=24.$

Transposing, $15+12-24=x.$

Changing terms, $x=15+12-24.$

Reducing, $x=3.$

24. Given $\frac{5x-4}{3}-\frac{x-3}{2}=13$ to find x.

Multiplying by 3, $5x-4-\frac{3x-9}{2}=39.$

Multiplying by 2, $10x-8-3x+9=78.$

Transposing, $10x-3x=78+8-9.$

Collecting terms, $7x=77.$

Dividing by 7, $x=11.$ *Ans.*

25. Given $\frac{mx-n}{a}=b$ to find x.

Multiplying by a, $mx-n=ab.$

Transposing, $mx=ab+n.$

Dividing by m, $x=\frac{ab+n}{m}.$ *Ans.*

IV.

147. Combining the foregoing rules and illustrations, we deduce the following

General Rule for solving all Simple Equations which contain only one unknown term :

1. *Clear the equation of fractions.*

2. *Transpose all the terms containing the unknown quantity to one side of the equation, and all the remaining terms to the other side, and reduce each member to its most simple form.*

3. *Divide each member of the equation by the coefficient of the unknown term.*

26. Given $2x-\frac{19}{4}=\frac{3x}{4}+4$ to find x.

Multiplying by 4, $8x-19=3x+16.$
Transposing, $8x-3x=16+19.$
Collecting, $5x=35.$
Dividing by 5, $x=7.$

27. Given $\frac{1-bx}{a}=\frac{1-ax}{b}$ to find x.

Multiplying by a, $1-bx=\frac{a-a^2x}{b}.$
Multiplying by b, $b-b^2x=a-a^2x.$
Transposing, $a^2x-b^2x=a-b.$
Dividing by a^2-b^2, $x=\frac{a-b}{a^2-b^2}=\frac{1}{a+b}.$

28. Given $\frac{a}{bx}+\frac{c}{dx}-\frac{a-c}{bdx}=h-\frac{1}{x}$ to find x.

Multiplying by bdx, $ad+bc-(a-c)=bdhx-bd.$
Omitting the parenthesis, $ad+bc-a+c=bdhx-bd.$
Changing and transposing, $bdhx=ad+bc+bd-a+c.$
Dividing by bdh, $x=\frac{ad+bc+bd-a+c}{bdh}.$

V.

148. If the terms of the equation contain both *simple* and *compound* denominators, it will, generally, be found convenient to divest it of the simple denominators at first, and afterwards of those which are compound.

29. Given $\frac{6x+7}{9}+\frac{7x+13}{6x+3}=\frac{2x+4}{3}$ to find x.

Multiplying by 9, $6x+7+\frac{63x+117}{6x+3}=6x+12.$

Transposing, $\frac{63x+117}{6x+3}=6x+12-6x-7=5.$

Multiplying by $6x+3$, $63x+117=30x+15.$

Transposing, $63x-30x=15-117.$

Reducing, $33x=-102.$

Dividing, $x=-3\frac{1}{11}.$

30. Given $\frac{2x+8\frac{1}{2}}{9}-\frac{13x-2}{17x-32}+\frac{x}{3}=\frac{7x}{12}-\frac{x+16}{36}$ to find x.

Multiplying all the terms by 36, it being the least common multiple of 9, 3, 12, and 36, we have

$$8x+34-\frac{468x-72}{17x-32}+12x=21x-x-16.$$

Reducing terms, $50=\frac{468x-72}{17x-32}.$

Multiplying by $17x-32$, $850x-1600=468x-72.$

Reducing terms, $382x=1528.$

Dividing by 382, $x=4.$ *Ans.*

EXAMPLES.

1. Given $5x+22-2x=31$ to find x. *Ans.* $x=3.$
2. Given $4-19x=14-21x$ to find x. *Ans.* $x=5.$
3. Given $24x-12=240-12x$ to find x. *Ans.* $x=7.$
4. Given $15x+7x-10=12x+90$ to find x. *Ans.* $x=10.$
5. Given $7x+2x=12x-36$ to find x. *Ans.* $x=12.$

6. Given $12x-3x-2x=63$ to find x. *Ans.* $x=9$.

7. Given $x+\frac{x}{4}+\frac{x}{5}=87$ to find x. *Ans.* $x=60$.

8. Given $x-\frac{x}{4}+13=\frac{x}{2}+40$ to find x. *Ans.* $x=108$.

9. Given $\frac{x}{5}+\frac{x}{12}=\frac{x}{10}+22$ to find x. *Ans.* $x=120$.

10. Given $x-\frac{x}{7}+20=\frac{x}{2}+\frac{x}{4}+26$ to find x. *Ans.* $x=56$.

11. Given $3x+\frac{3x}{4}+15=\frac{x}{2}+41$ to find x. *Ans.* $x=8$.

12. Given $x-\frac{4x+8}{6}=8$ to find x. *Ans.* $x=28$.

13. Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$ to find x.
Ans. $x=9$.

14. Given $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$ to find x. *Ans.* $x=5$.

15. Given $17x-\frac{5x-4}{3}-\frac{8x+4}{5}=20x-\frac{3x+8}{2}-5$ to find x.
Ans. $x=2$.

16. Given $9x-\frac{x-1}{2}+\frac{2x-2}{3}=12x-\frac{5x-7}{4}-13$ to find x.
Ans. $x=7$.

17. Given $x+\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=2x+17$ to find x.
Ans. $x=60$.

18. Given $\frac{a}{x}=b+c$ to find x. *Ans.* $x=\frac{a}{b+c}$.

19. Given $8x-40=0$ to find x. *Ans.* $x=5$.

20. Given $a+\frac{1}{x}=b+c+\frac{d}{x}$ to find x. *Ans.* $x=\frac{d-1}{a-b-c}$.

21. Given $x-\frac{3x-3}{5}+4=\frac{20-x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5}$ to find the value of x. *Ans.* $x=6$.

22. Given $ax^2+bx=mx^2+nx$ to find x. *Ans.* $x=\frac{n-b}{a-m}$.

23. Given $ax+m=bx+n$ to find x. *Ans.* $x=\frac{n-m}{a-b}$.

24. Given $\frac{3x}{b}-\frac{x}{c}=m-c$ to find x. *Ans.* $x=\frac{bc(m-c)}{3c-b}$.

25. Given $\frac{7x}{a}-\frac{3}{m}=15x+n$ to find x. *Ans.* $x=\frac{amn+3a}{7m-15am}$.

26. Given $\frac{a-x}{b}-\frac{4a-x}{c}=a-b$ to find x.

Ans. $x=\frac{4ab-ac+abc-b^2c}{b-c}$.

27. Given $\frac{a^2x}{b-c}+de=3x-\frac{d}{e}$ to find x.

Ans. $x=\frac{cde^2-bde^2-bd+cd}{3ce+a^2e-3be}$.

28. Given $5x-\frac{4x-a}{b}+\frac{2x+2a}{4}=m+n-\frac{2x+2a}{c}$ to find x.

Ans. $x=\frac{2bcm+2bcn-4ab-2ac-abc}{11bc-8c+4b}$.

29. Given $\frac{6x+18}{13}-4\frac{5}{6}-\frac{11-3x}{36}=5x-48-\frac{13-x}{12}-\frac{21-2x}{18}$ to find the value of x. *Ans.* $x=10$.

30. Given $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$ to find the value of x.

Ans. $x=6$.

SECTION VIII.

PROBLEMS.

1. A gentleman stated that his age was twice that of his oldest son, and that the sum of their ages was 72 years. Required the age of each.

Let $x =$ the age of the son.
Then $2x =$ the age of the gentleman.
Therefore, $x+2x=72$, the age of both.
Or, $3x=72$.
Dividing, $x=24$, the age of the son.
$2x=48$, the age of the gentleman.
Proof, $24+48=72$.

2. What number is that, to which if $\frac{4}{7}$ of it be added, the sum will be 99?

Let $x =$ the number.
Then, $\frac{4x}{7}+x=99$.
Clearing of fractions, $4x+7x=693$.
Collecting the terms, $11x=693$.
Dividing, $x=63$, the number.

3. A's and B's estates are valued at $3240, but B's is only $\frac{7}{8}$ the value of A's. What is the property of each?

Let $x =$ A's estate.
Then, $\frac{7x}{8} =$ B's estate.
Therefore, $x+\frac{7x}{8}=3240$.
Clearing of fractions, $8x+7x=25920$.
Or, $15x=25920$.
Dividing, $x=1728$, A.'s estate.
$\frac{7\times1728}{8}=1512$, B.'s estate.

4. If $\frac{2}{3}$ of a certain number be added to $\frac{1}{2}$ of it, the sum will be 98. Required the number.

Let $x =$ the number.
Then, $\frac{2x}{3}+\frac{x}{2}=98$.
Clearing of fractions, $4x+3x=588$.
Or, $7x=588$.
Dividing, $x=84$. *Ans.*

5. A certain gentleman divided his property, consisting of \$5300, among his four sons, A, B, C, and D. He gave \$350 more to B than A; he gave C \$400 more than B; but he gave D twice as much as he gave A and B. How much did each son receive?

Let x = A's share.
Then $x+350$ = B's share.
And $x+350+400$ = C's share.
And $2(2x+350)=4x+700=$ D's share.
Therefore, $x+x+350+x+350+400+4x+700=5300$
Reducing, $7x+1800=5300.$
Transposing, $7x=5300-1800.$
Reducing, $7x=3500.$
Dividing, $x=500$, A's share.
$500+350=850$, B's share.
$850+400=1250$, C's share.
$2(500+850)=2700$, D's share.

Verification, $500+850+1250+2700=5300.$

6. Divide \$70 among James, John, and Charles; give John twice as much as James, and give Charles twice as much as John.

Let x = the sum given to James.
Then $2x$ = the sum given to John.
And $4x$ = the sum given to Charles.
Then, by the conditions of the question,

$$x+2x+4x=70.$$

Or, $7x=70.$
Dividing, $x=10$, the sum given James.
$2x=20$, the sum given John.
$4x=40$, the sum given Charles.
Verification, $10+20+40=70.$

7. Two men found a purse containing \$144, and it was agreed that B should have \$30 more than A. How many dollars did each receive?

Let x = the sum A received.
Then $x+30$ = the sum B received.
Therefore, $x+x+30=144$.
Or, $2x+30=144$.
Transposing, $2x=144-30$.
Or, $2x=114$.
Dividing, $x=57$, the sum A received.
$x+30=87$, the sum B received.
Verification, $57+87=144$.

8. My horse and chaise are worth $336, but the chaise is worth five times as much as the horse. What is the value of each?

Let x = the value of the horse.
Then $5x$ = the value of the chaise.
And, $x+5x=336$.
Or, $6x=336$.
Dividing, $x=56$ = value of the horse.
$5x=280$ = value of the chaise.
Proof, $56+280=336$.

9. What number is that whose third part exceeds its fifth by 12?

Let x = the number required.
Then its third part will be $\frac{x}{3}$.
And its fifth part, $\frac{x}{5}$.
Therefore, $\frac{x}{3}-\frac{x}{5}=12$.
Multiplying all the terms by 15, we have,
$$5x-3x=180.$$
Or, $2x=180$.
Dividing, $x=90$, the number required.

10. John Smith's oldest daughter is 15 years old, and his youngest daughter is 11; he has $1728, which he wishes to give them. How shall he divide this sum, that each may deposit her share in a bank which pays 6 per cent. simple interest,

so that both shall have an equal sum when they are 21 years old?

Let $x =$ the sum the youngest receives.

And, $1728-x =$ the sum the oldest receives.

Then, $x+x\times.06\times10=1728-x+(1728-x\times.06\times6)$.

Or, $x+.6x=1728-x+622.08-.36x$.

Transposing, $2.96x=2350.08$.

Dividing, $x=\$793\frac{35}{37}$, the youngest receives.

$\$1728-\$793\frac{35}{37}=\$934\frac{2}{37}$, the oldest receives.

Let the pupil prove this question.

11. A man being asked the value of his horse and saddle, replied that his horse was worth \$114 more than his saddle, and that $\frac{2}{3}$ the value of the horse was 7 times the value of his saddle. What was the value of each?

Let $x =$ the value of the saddle.

And $x+114 =$ the value of the horse.

Then, $\frac{2}{3}(x+114)=7x$.

Or, $2x+228=21x$

Transposing, $19x=228$.

Dividing, $x=\$12$, value of the saddle.

$\$12+\$114=\$126$, value of the horse.

12. A can reap a field in 7 days, B can reap it in 5 days. In what time can they both reap it together?

Let $x =$ the days they would reap it together.

A would reap $\frac{1}{7}$ of it in a day, and B would reap $\frac{1}{5}$ of it in a day; therefore in one day both together would reap $\frac{1}{7}+\frac{1}{5}=\frac{12}{35}$ of it.

But, by the conditions, the field was to be reaped in x days.

Therefore, $\frac{12}{35} : 1 :: 1 \text{ day} : x \text{ days}$.

Multiplying extremes, $\frac{12x}{35}= 1$.

Multiplying by 35, $12x=35$.

Dividing, $x= 2\frac{11}{12}$ days. *Ans.*

13. I have two carriages; the value of one is five times that of the other, and the value of my horse is equal to both of my

carriages. The worth of them all is $300. What is the value of each?

Ans. First carriage $25, second carriage $125, horse $150.

14. A gentleman being asked his age, replied that his was twice that of his wife, and that his wife was three times as old as his daughter, and that the sum of their ages was 120 years. Required the age of each.

Ans. { Gentleman's age, 72 years.
His wife's age, 36 years.
His daughter's age, 12 years.

15. A man met 4 beggars, to whom he gave 77 cents. To the first he gave twice as many as to the second; to the third, as many as he gave to the first and second; and to the fourth, as many as he gave to the first and third. What sum did he give each?

Ans. First 14 cents, second 7, third 21, fourth 35.

16. A drover has a lot of oxen and cows, for which he gave $1428. For the oxen he gave $55 each, and for the cows $32 each, and he had twice as many cows as oxen. Required the number of each. *Ans.* 12 oxen, 24 cows.

17. A gentleman, at his decease, left an estate of $1872 for his wife, three sons, and two daughters. His wife was to receive three times as much as either of her daughters, and his sons to receive each one half as much as one of the daughters. Required the sum each received.

Ans. Wife $864, daughters $288 each, sons $144 each.

18. A boy bought apples, oranges, and pears; he gave two cents a-piece for the apples, three cents for the oranges, and four cents for the pears. He had twice as many oranges as apples, and three times as many pears as oranges. The sum he expended was $2.24. How many did he buy of each kind?

Ans. 7 apples, 14 oranges, 42 pears.

19 Let 85 be divided into two such parts that one of them shall be four times as large as the other. *Ans.* 17 and 68.

20. Divide $100 among A, B, and C, so that A may have $20 more than B, and B $10 more than C.

Ans. A $50, B $30, and C $20.

21. A prize of $1000 is to be divided between A and B, so that their shares may be in the proportion of 7 to 8; required the share of each. *Ans.* A's share $466⅔, and B's $533⅓.

22. What number is that whose 3d part exceeds its 5th part by 6⅖? *Ans.* 48.

23. A laborer agreed to serve for 36 days on these conditions; that for every day he worked he was to receive $1.25, but for every day he was absent he was to forfeit $0.50. At the end of the time he received $17. It is required to find how many days he labored, and how many days he was absent.

Ans. He labored 20 days, and was absent 16 days.

24. Out of a cask of wine, which had leaked away ⅓, 13 gallons were drawn, and then being gauged it was found to be half full. How many gallons did the cask contain?

Ans. 78 gallons.

25. Divide 30 into two such parts that ⅔ of the one shall exceed 4⁄9 of the other by 6⅔. *Ans.* 18 and 12.

26. What two numbers are those whose difference is 3, and the difference of whose squares is 51? *Ans.* 10 and 7.

27. Three men, A, B, and C, trade in company; A put in a certain sum, B put in twice as much as A, and C put in three times as much as both, and they gain $864. What is each man's share of the gain?

Ans. A's $72, B's $144, C's $648.

28. James and William have between them 44 apples, and James says to William, if you will give me 12 of your apples, your number will then be only 2⁄9 of mine. William replied, if you will give me 12 of yours, your number will then be only ⅜ of mine. Required the number of each.

Ans. James had 24 apples, and William 20.

29. Let 112 be divided into two such numbers that the greater shall be to the less as 9 to 7. *Ans.* 63 and 49.

30. Let 19 be divided into two such parts that three times the greater shall be equal to four times the less. Required those numbers. *Ans.* $10\frac{6}{7}$ and $8\frac{1}{7}$.

31. There are two numbers whose sum is 24, and if 7 be added to the larger, and 4 to the less, their ratio will be as 4 to 3. Required those numbers. *Ans.* 13 and 11.

32. The difference of two numbers is 4, and 7 times the larger number is equal to 11 times the less. Required those numbers. *Ans.* 11 and 7.

33. A merchant has two kinds of grain, one at $2.50 per bushel, and the other at $2 per bushel. He wishes to make a mixture of 80 bushels, that shall be worth $2.10 per bushel. How many bushels of each sort must he use?

Ans. 16 bushels at $2.50, and 64 at $2.

34. A man having lost $\frac{1}{4}$ of his money, found he had $96 left. Required the sum he had at first. *Ans.* $128.

35. J. Jones found a certain sum of money, which was equal to $\frac{1}{2}$ of what he possessed; but having spent $40, the remainder was $\frac{4}{5}$ of the sum he found. Required the sum he at first possessed. *Ans.* $\$36\frac{4}{11}$.

36. In my school $\frac{2}{5}$ of my pupils study grammar, $\frac{2}{3}$ of the remainder read, 10 spell, and the remainder, which is $\frac{1}{7}$ of the number that read, study navigation. Required the number of pupils in the school. *Ans.* 70 pupils.

37. A gentleman lent a certain sum of money for 3 years at 5 per cent. compound interest; that is, at the end of each year he added $\frac{1}{20}$ to the sum due. At the close of the third year he lost $15.25, but then there remained due to him $2300. Required the sum lent. *Ans.* $2000.

38. A spendthrift spent $\frac{1}{5}$ of the fortune left him by his father, and he then earned $124. Soon after he lost in speculation $\frac{2}{3}$ of his property, after which he gained $274. His

property was now valued at $\frac{1}{2}$, wanting \$86, of his original estate. What was the sum left him by his father?

Ans. \$1720.

39. A asked B how much money he had. He replied, if I had 5 times the sum I now possess I could lend you \$60, and then $\frac{1}{5}$ of the remainder would be equal to $\frac{1}{2}$ the dollars I now have. Required the sum which B had. *Ans.* \$24.

40. A gentleman left an estate of \$1862 for his three sons. He gave his youngest \$133 less than his second son, and to his oldest son he gave as much as to the other two. How much did each receive?

Ans. Youngest son \$399, second \$532, oldest \$931.

41. A, B and C, found a purse of money, and it was mutually agreed that A should receive \$15 less than one-half, and that B should have \$13 more than one quarter, and that C should have the remainder, which was \$27. How many dollars did the purse contain? *Ans.* \$100.

42. Lent my good friend S. Jenkins a certain sum of money, at 6 per cent., which he kept until the interest was $\frac{3}{7}$ of the principal. The sum then due was \$500. Required the sum lent. *Ans.* \$350.

43. A certain man added to his estate $\frac{1}{4}$ its value, and then lost \$760. But he afterwards gained \$600. His property then amounted to \$2000. What was the value of his estate at first?

Ans. \$1728.

44. James said to John, I have 40 shillings more than you. Yes, replied the other, and $\frac{1}{9}$ of yours is equal to $\frac{1}{4}$ of mine. Required the number of shillings that each had.

Ans. James 72 shillings, and John 32.

45. A merchant bought a number of barrels of flour, and having sold half the number and 4 barrels more to A, and $\frac{3}{4}$ of the remainder wanting 4 barrels to B, he had 20 barrels remaining. Required the number the merchant bought.

Ans. 136 barrels.

46. What number is that from which, if 7 be subtracted, $\frac{1}{6}$ of the remainder will be 5? *Ans.* 37.

47. It is required to divide 44 into two such numbers that $\frac{3}{4}$ of one of them shall be 6 more than $\frac{3}{5}$ of the other.

Ans. 24 and 20.

48. It is required to divide the number 43 into two such parts that one of them shall be 3 times as much above 20 as the other wants of 17. Required the numbers.

Ans. 29 and 14.

49. John Jones can reap a certain field in 10 days, but, with the help of his oldest son, he can do it in 8 days. How long would it require his son to perform the labor himself?

Ans. 40 days.

50. A engaged to reap a field for 90 shillings, and he could perform the labor in 9 days; but he took in B as a partner, and they supposed it would require 5 days for both to perform the labor, but they finished it in 4 days. How much, in justice, must A pay to B? *Ans.* 50 shillings.

51. I have two horses, and a saddle worth $30. Now, the saddle and first horse are worth $\frac{3}{5}$ the second horse, but the saddle and second horse are worth three times the first horse. Required the value of each.

Ans. First horse $60, second horse $150.

52. A gentleman let $\frac{3}{8}$ of his money at 5 per cent., and the remainder at 6 per cent., and his interest amounted to $180. What were the sums lent?

Ans. $1200 at 5 per cent., $2000 at 6 per cent.

53. A can do a piece of work in 12 days, B can do the same work in 10 days, and C can perform it in 8 days. How long would it require A and B to do it; how long A and C; how long B and C; and how long A, B and C, to perform the labor?

Ans. A and B $5\frac{5}{11}$ days, A and C $4\frac{4}{5}$ days, B and C $4\frac{4}{9}$ days, A, B and C, $3\frac{9}{37}$ days.

54. Lent \$780, at 6 per cent., for 5 years. What principal will amount to the sum in 4 years, at 10 per cent?

Ans. \$724.28$\frac{4}{7}$.

55. Lent my neighbor Jenkins \$270 for 4 years, at 6 per cent.; some time afterwards, I borrowed of him \$500, at 8 per cent. How long shall I keep it, to balance the favor?

Ans. 1$\frac{31}{50}$ years.

56. A fox is pursued by a greyhound, and is 60 of her own leaps before him. The fox makes 9 leaps while the greyhound makes but 6; but the latter in 3 leaps goes as far as the former in 7. How many leaps does the greyhound make before he catches the fox?

Ans. The greyhound makes 72 leaps, and the fox 108.

57. A gentleman gave in charity \$46; a part thereof in equal portions to five poor men, and the rest in equal portions to 7 poor women. Now, a man and a woman had between them \$8. What was given to the men, and what to the women?

Ans. The men received \$25, and the women \$21.

58. A man has two farms, and his stock is worth \$183. Now, the stock and his first farm is worth once and two-sevenths the value of the second farm, but the stock and the second farm is worth once and five-eighths the value of the first farm. What is the value of each farm?

Ans. First farm, \$384; second farm, \$441.

59. A certain clock has an hour hand, a minute hand, and a second hand, all turning on the same centre. At 12 o'clock all the hands are together, and point at 12. How long will it be before the second hand will be between the other two hands, and at equal distances from each? Also, before the minute hand will be equally distant between the other two hands? Also, before the hour hand will be equally distant between the other two hands?

Ans. 60$\frac{780}{1427}$ seconds, 61$\frac{683}{697}$ seconds, 59$\frac{13}{73}$ seconds.

60. What number is that, the treble of which, increased by 12, shall as much exceed 54 as that treble is less than 144?

Ans. 31.

SECTION IX.

EQUATIONS OF THE FIRST DEGREE, CONTAINING TWO UNKNOWN QUANTITIES.

Art. 149. When the problem contains two unknown quantities, there must be two independent equations involving them; and from them an equation may be deduced, which shall contain only one of the unknown quantities.

The process by which one of the unknown quantities is thus removed is called elimination; and this may be performed in three ways.

First, by Addition and Subtraction.
Second, by Comparison.
Third, by Substitution.

150. Elimination by addition and subtraction.

EXAMPLES.

1. Given $\left\{\begin{matrix} 3x-2y=11 \\ 6x+5y=67 \end{matrix}\right\}$ to find the value of x and y.

1. By first condition, $3x-2y=11$.
2. By second " $6x+5y=67$.
3. Multiplying 1st by 2, $6x-4y=22$.
4. Multiplying 2d by 1, $6x+5y=67$.
5. Subtracting 3d from 4th, $9y=45$.
6. Dividing 5th by 9, $y=5$.
7. Multiplying 1st by 5, $15x-10y=55$.
8. Multiplying 2d by 2, $12x+10y=134$.
9. Adding 7th and 8th, $27x=189$.
10. Dividing 9th by 27, $x=7$.

VERIFICATION.

$$3\times7-2\times5=21-10=11.$$
$$6\times7+5\times5=42+25=67.$$

2. Given $\left\{\begin{matrix} 5x+4y=23 \\ 6x-3y=12 \end{matrix}\right\}$ to find the value of x and y.

1. By the first condition, $5x+4y=23$.
2. By the second, $6x-3y=12$.
3. Multiplying 1st by 6, $30x+24y=138$.
4. Multiplying 2d by 5, $30x-15y=60$.
5. Subtracting 4th from 3d, $39y=78$.
6. Dividing 5th by 39, $y=2$.
7. Multiplying 1st by 3, $15x+12y=69$.
8. Multiplying 2d by 4, $24x-12y=48$.
9. Adding 7th and 8th, $39x=117$.
10. Dividing 9th by 39, $x=3$.

VERIFICATION.

$$5\times3+4\times2=15+8=23.$$
$$6\times3-3\times2=18-6=12.$$

3. A says to B, if $\frac{1}{5}$ of my age were added to $\frac{2}{3}$ of yours, the sum would be $19\frac{1}{3}$ years. But, says B, if $\frac{2}{5}$ of mine were subtracted from $\frac{7}{8}$ of yours, the remainder would be $18\frac{1}{4}$ years. Required the sum of their ages.

1. By first condition, $\frac{x}{5}+\frac{2y}{3}=19\frac{1}{3}$.
2. By the second, $\frac{7x}{8}-\frac{2y}{5}=18\frac{1}{4}$.
3. Clearing the 1st of fractions, $3x+10y=290$.
4. Clearing the 2d, $35x-16y=730$.
5. Multiplying 3d by 35, $105x+350y=10150$.
6. Multiplying 4th by 3, $105x-48y=2190$.
7. Subtracting 6th from 5th, $398y=7960$.
8. Dividing 7th by 398, $y=20$.
9. Substituting 20 for y in the 3d, $3x+200=290$.
10. Transposing and uniting, $3x=90$.
11. Dividing 10th by 3, $x=30$.

VERIFICATION.

$$\frac{30}{5}+\frac{2\times20}{3}=6+13\tfrac{1}{3}=19\tfrac{1}{3}.$$
$$\frac{7\times30}{8}-\frac{2\times20}{5}=26\tfrac{1}{4}-8=18\tfrac{1}{4}.$$

From the operation of the preceding examples, we deduce the following

RULE. *Multiply or divide the given equations by such numbers or quantities as will make the term that contains one of the unknown quantities the same in each of them; then add or subtract the two equations thus obtained, and there will arise a new equation with only one unknown quantity in it, which may be resolved by* Art. 147.

151. ELIMINATION BY COMPARISON.

4. Given $\left\{\begin{array}{l}2x+3y=17\\5x-2y=14\end{array}\right\}$ to find the values of x and y.

1. By the first condition, $2x+3y=17$.
2. By the second, $5x-2y=14$.
3. Transposition of the 1st, $2x=17-3y$.
4. Dividing the 3d by 2, $x=\frac{17-3y}{2}$.
5. Transposition of the 2d, $5x=14+2y$.
6. Dividing the 5th by 5, $x=\frac{14+2y}{5}$.

As things which are equal to the same are equal to each other, we therefore infer that $\frac{17-3y}{2}$, in the 4th, is equal to $\frac{14+2y}{5}$ in the 6th; because they are both equal to x.

7. Therefore, $\frac{17-3y}{2}=\frac{14+2y}{5}$.
8. Clearing of fractions, $85-15y=28+4y$.
9. Transposing 8th, $19y=57$.
10. Dividing 9th by 19, $y=3$.
11. Substituting 3 for the value of y in the first equation, we have, $2x=17-9$.
12. By reduction, $2x=8$.
13. Dividing 12th by 2, $x=4$.

VERIFICATION.

$$2\times4+3\times3=8+9=17.$$
$$5\times4-2\times3=20-6=14.$$

Hence the following

Rule. *Observe which of the unknown quantities is least involved, and find its value in each of the equations, as in* Art. 148.

Let the two values thus found be made equal to each other, and there will arise a new equation, with only one unknown quantity in it, whose value may be found as in Art. 147.

152. Elimination by substitution.

5. Two boys playing marbles, the older said to the younger, if you had three times as many marbles as you now possess, the sum of yours and mine would be 19. But the younger replied, if twice the number of mine were subtracted from four times as many as you have, the number would be 20. Required the number of marbles that each possessed.

Let x represent the marbles of the elder;
And y the number of the younger.

1. Then, by the condition of the question, $x+3y=19.$
2. And $4x-2y=20.$
3. Transposing the 1st, $x=19-3y$
4. Putting the 3d into the 2d, $4(19-3y)-2y=20.$
5. Then, $76-12y-2y=20.$
6. Transposing and reducing, $y=4.$
7. Putting the value of y into the 1st, $x+12=19.$
8. Transposing and reducing, $x=19-12=7.$

Ans. The elder had 7 marbles, and the younger 4.

VERIFICATION.

$$7+3\times4=7+12=19.$$
$$4\times7-2\times4=28-8=20.$$

By the above method of operation, we deduce the following

Rule. *Find the value of either of the unknown quantities in that equation in which it is least involved; then substitute this value in the place of its equal in the other equation, and there* *will arise a new equation, with only one unknown quantity in it ; the value of which may be found by* Art. 147.

EXAMPLES.

6. Given $\left\{\begin{array}{l}3x+7y=33\\2x+4y=20\end{array}\right\}$ Required x and y.

Ans. $x=4$; $y=3$.

7. Given $\left\{\begin{array}{l}7x+2y=39\\3x-4y=\ \ 7\end{array}\right\}$ Required x and y.

Ans. $x=5$; $y=2$.

8. Given $\left\{\begin{array}{l}6x-3y=\ \ 27\\4x-6y=-2\end{array}\right\}$ Required x and y.

Ans. $x=7$; $y=5$.

9. Given $\left\{\begin{array}{l}7x+3y=62\\5x-2y=36\end{array}\right\}$ Required x and y.

Ans. $x=8$; $y=2$.

10. Given $\left\{\begin{array}{r}12x+8y=116\\2x-\ \ y=\ \ \ \ 3\end{array}\right\}$ Required x and y.

Ans. $x=5$; $y=7$.

11. Given $\left\{\begin{array}{r}11x+3y=\ \ 124\\2x-6y=-56\end{array}\right\}$ to find x and y.

Ans. $x=8$; $y=12$.

12. Given $\left\{\begin{array}{l}9x+4y=58\\3x+2y=26\end{array}\right\}$ to find x and y.

Ans. $x=2$; $y=10$.

13. Given $\left\{\begin{array}{l}6x+5y=112\\8x-2y=\ \ 80\end{array}\right\}$ to find the value of x and y.

Ans. $x=12$; $y=8$.

14. Given $\left\{\begin{array}{l}7x-2y=-6\\2x+2y=\ \ 24\end{array}\right\}$ to find the value of x and y.

Ans. $x=2$; $y=10$.

15. Given $\left\{\begin{array}{l}6x+11y=\ \ 115\\8x-22y=-30\end{array}\right\}$ to find the value of x and y.

Ans. $x=10$; $y=5$.

16. Given $\left\{\begin{array}{r}2x+\ 3y=\ \ \ \ 47\\10x-12y=-62\end{array}\right\}$ to find the value of x and y.

Ans. $x=7$; $y=11$.

17. Given $\left\{\begin{array}{l} \frac{x}{2}-\frac{y}{3}=0 \\ \frac{x}{4}+\frac{y}{6}=6 \end{array}\right\}$ to find the value of x and y.

Ans. $x=12$; $y=18$.

18. Given $\left\{\begin{array}{r} \frac{3x}{5}-y=11 \\ x+\frac{y}{5}=37 \end{array}\right\}$ to find the value of x and y.

Ans. $x=35$; $y=10$.

19. Given $\left\{\begin{array}{r} \frac{4x}{7}-\frac{2y}{3}=\ \ 2 \\ x+7y=175 \end{array}\right\}$ to find the value of x and y.

Ans. $x=28$; $y=21$.

20. Given $\left\{\begin{array}{r} \frac{7x}{8}-\frac{y}{9}=\ 19 \\ 3x+3y=126 \end{array}\right\}$ to find the value of x and y.

Ans. $x=24$; $y=18$.

21. Given $\left\{\begin{array}{r} 14x+\frac{5y}{6}=\ 38 \\ x+12y=146 \end{array}\right\}$ to find the value of x and y.

Ans. $x=2$; $y=12$.

22. Given $\left\{\begin{array}{l} \frac{x}{7}-\frac{7y}{10}=-20 \\ \frac{x}{4}+3y=134 \end{array}\right\}$ to find the value of x and y.

Ans. $x=56$; $y=40$.

23. Given $\left\{\begin{array}{l} ax+by=c \\ mx+ny=d \end{array}\right\}$ to find the value of x and y.

Ans. $x=\frac{bd-nc}{bm-an}$; $y=\frac{ad-mc}{an-bm}$.

24. Given $\left\{\begin{array}{l} \frac{x}{a}-\frac{y}{b}=m \\ \frac{x}{c}+\frac{y}{d}=n \end{array}\right\}$ to find x and y.

Ans. $x=\frac{abcm+acdn}{ad+bc}$; $y=\frac{bcdn-abdm}{ad+bc}$.

25. Given $\left\{\begin{aligned} &\frac{x}{2}-12=\frac{y}{4}+8 \\ &\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27 \end{aligned}\right\}$ to find the value of x and y.

Ans. $x=60$; $y=40$.

SECTION X.

ELIMINATION WHERE THERE ARE THREE OR MORE UNKNOWN QUANTITIES INVOLVED IN AN EQUAL NUMBER OF EQUATIONS.

Rule. *Find the values of one of the unknown quantities in each of the three given equations, as if all the others were known; then put the first of these values equal to the second, and either the first or second equal to the third, and there will arise two new equations with only two unknown quantities in them, the values of which may be found as in* Art. 147, *and thence the value of the third.*

Or, the unknown quantities may be obtained by multiplying each of the three equations by such quantities as will make one of their terms the same in all of them; then, having subtracted any two of these resulting equations from the third, or added them together, as the case may require, there will remain only two equations, which may be resolved by the former rules.

Or, we may find the value of one of the unknown quantities in that equation in which it is least involved, and then substitute this value for that unknown quantity in all the other equations, and, proceeding in the same way with these equations, we obtain the other unknown quantities.

EXAMPLES.

1. Given $\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\left\{\begin{aligned} &x+\ y+2z=41 \\ &x+3y+\ z=47 \\ &\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10 \end{aligned}\right\}$ to find the value of x, y and z.

4. From the 1st equation, $x=41-y-2z.$

5. From the 2d, $x=47-3y-z.$

6. From the 3d, $x=20-\frac{2y}{3}-\frac{z}{2}.$

7. Equal values of x in 4th and 5th, $41-y-2z=47-3y-z.$

8. Value of y in 7th, $y=\frac{6+z}{2}.$

9. Equal values of x in 4th and 6th, $41-y-2z=20-\frac{2y}{3}-\frac{z}{2}.$

10. Value of y in 9th, $y=63-\frac{9z}{2}.$

11. Equal values of y in 8th and 10th, $\frac{6+z}{2}=63-\frac{9z}{2}.$

12. Reducing, $z=12.$

13. Substituting for z its value in 8th, $y=\frac{6+12}{2}=9.$

14. Substituting for y and z their values in 4th, $x=41-9-24=8.$

2. Given $\begin{matrix}1\\2\\3\end{matrix}\left\{\begin{aligned}5x+4y-2z&=28\\10x-6y+4z&=30\\2x+y-z&=9\end{aligned}\right\}$ to find the value of x, y, and z.

Subtracting the 2d from twice the 1st, we have,

4. $14y-8z=26.$

Subtracting the 2d from 5 times the 3d,

5. $11y-9z=15.$

Subtracting 14 times the 5th from 11 times the 4th,

6. $38z=76.$

7. $z=2.$

Substituting for z its value in the 5th,

8. $11y-18=15.$

9. $y=3.$

Substituting for y and z their values in the 3d,

10. $2x+3-2=9.$

11. $x=4.$

3. Given $\left\{\begin{array}{l} 3x-y-2z=0 \\ 6x+2y+3z=45 \\ 4x+3y-z=31 \end{array}\right\}$ to find x, y, and z.

Ans. $x=4$; $y=6$; $z=3$.

4. Given $\left\{\begin{array}{l} 8x-9y-7z=-36 \\ 12x-y-3z=36 \\ 6x-2y-z=10 \end{array}\right\}$ to find x, y, and z.

Ans. $x=4$; $y=6$; $z=2$.

5. Given $\left\{\begin{array}{l} 7x+4y-z=78 \\ 4x-5y-3z=-21 \\ x-3y-4z=-37 \end{array}\right\}$ to find x, y, and z.

Ans. $x=8$; $y=7$; $z=6$.

6. Given $\left\{\begin{array}{l} x+y=30 \\ x+z=25 \\ y+z=15 \end{array}\right\}$ to find x, y, and z.

Ans. $x=20$; $y=10$; $z=5$.

7. Given $\left\{\begin{array}{l} 8x-4y=24-z \\ 6x+y=z+84 \\ x+80=3y+4z \end{array}\right\}$ to find x, y, and z.

Ans. $x=12$; $y=20$; $z=8$.

8. Given $\left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}-\frac{z}{4}=23 \\ \frac{x}{3}-\frac{y}{4}+\frac{z}{2}=12 \\ \frac{x}{4}+\frac{y}{2}-\frac{z}{3}=17 \end{array}\right\}$ to find x, y, and z.

Ans. $x=36$; $y=24$; $z=12$.

9. Given $\left\{\begin{array}{l} 3u+x+2y-z=22 \\ 4x-y+3z=35 \\ 4u+3x-2y=19 \\ 2u+4y+2z=46 \end{array}\right\}$ to find u, x, y, and z.

Ans. $u=4$; $x=5$; $y=6$; $z=7$.

EQUATIONS OF THE FIRST DEGREE, CONTAINING SEVERAL UNKNOWN QUANTITIES.

EXAMPLES.

1. A says to B and C, give me half of your money, and I shall have \$55. B replies, if you two will give me one third of yours, I shall have \$50. But C says to A and B, if I had one fifth of your money, I should have \$50. Required the sum that each possessed.

Ans. A=\$20, B=\$30, C=\$40.

2. A merchant has three kinds of sugar. He can sell 3 lbs. of the first quality, 4 lbs. of the second quality, and 2 lbs. of the third quality, for 60 cents; or, he can sell 4 lbs. of the first quality, 1 lb. of the second quality, and 5 lbs. of the third quality, for 59 cents; or, he can sell 1 lb. of the first quality, 10 lbs. of the second quality, and 3 lbs. of the third quality, for 90 cents. Required the price of each quality.

Ans. First quality, 8 cents per lb.; second, 7 cents; third, 4 cents.

3. A gentleman's two horses, with their harness, cost him \$120. The value of the worst horse, with the harness, was double that of the best horse; and the value of the best horse, with the harness, was triple that of the worst horse. What was the value of each?

Ans. Harness, \$50; best horse, \$40; worst, \$30.

4. Find three numbers, so that the first with half the other two, the second with one third of the other two, and the third with one fourth of the other two, shall each be equal to 34.

Ans. 10, 22, and 26.

5. Find a number of three places, of which the digits have equal differences in their order; and, if the number be divided by half the sum of the digits, the quotient will be 41; and, if 396 be added to the number, the digits will be inverted.

Ans. 246.

6. A farmer has a large box, filled with wheat and rye; seven times the bushels of wheat is equal to four times the bushels of rye, wanting 3 bushels; and the quantity of wheat is to the quantity of rye as 3 to 5. Required the bushels of wheat and the bushels of rye.

Ans. Wheat 9 bushels, rye 15 bushels.

7. A says to B, if 7 times my property were added to $\frac{1}{7}$ of yours, the sum would be \$990. B replied, if 7 times my property were added to $\frac{1}{7}$ of yours, the sum would be \$510. Required the property of each. *Ans.* A's, \$140; B's, \$70.

8. If $\frac{1}{7}$ of A's age were subtracted from B's age, and 5 years added to the remainder, the sum would be 6 years; and if four years were added to $\frac{1}{5}$ of B's age, it would be equal to $\frac{1}{14}$ of A's age. Required their ages.

Ans. A's, 98 years; B's, 15 years.

9. What fraction is that, if 1 be added to its numerator, its value is $\frac{1}{3}$; or, if 1 be added to its denominator, its value is $\frac{1}{4}$?

Ans. $\frac{4}{15}$.

10. A says to B, if $\frac{1}{2}$ the difference of our ages were subtracted from my age, the remainder would be 25 years. B replies, if $\frac{1}{5}$ of the sum of our ages were taken from mine, the remainder would be $\frac{1}{3}$ of yours. Required their ages.

Ans. A's, 30 years; B's, 20 years.

11. There are two numbers, and if $\frac{1}{4}$ of their difference were taken from 4 times their sum, the remainder would be 62; but the difference of their sum and difference is equal to $\frac{2}{3}$ of the larger number. Required the numbers. *Ans.* 12 and 4.

12. Three men reckoning their money, says the first, if \$100 were added to my money, it would be as much as you both possess. Says the second, if \$100 were added to my money, I should have twice as much as you two have. Says the third man, if \$100 were added to mine, I should have three times as much as you both have. How much money had each man?

Ans. First, \$$9\frac{1}{11}$, second, \$$45\frac{5}{11}$, third, \$$63\frac{7}{11}$.

13. A, B and C, speaking of their ages, A said that the sum

of their ages was 90. B replied, that if his age were taken from the sum of the other two, the remainder would be 30. C said, if his age were taken from the other two, the remainder would be $\frac{1}{4}$ his age. Required their ages.

Ans. A's, 20; B's, 30; C's, 40.

14. There are 4 men, A, B, C and D, the value of whose estate is $14,000; twice A's, three times B's, half of C's, and one fifth of D's, is $16,000; A's, twice B's, twice C's, and two fifths of D's, is $18,000; and half of A's, with one third of B's, one fourth of C's, and one fifth of D's, $4000. Required the property of each.

Ans. A's, $2000; B's, $3000; C's, $4000; D's, $5000.

15. Find four numbers, such that the first, together with half the second, may be 357; the second, with $\frac{1}{3}$ of the third, equal to 476; the third, with $\frac{1}{4}$ of the fourth, equal to 595; and the fourth, with $\frac{1}{5}$ of the first, equal to 714.

Ans. First number, 190; second, 334; third, 426; fourth, 676.

16. If I were to enlarge my field by making it 5 rods longer and 4 rods wider, it would contain 240 square rods more than it now does; but, if I were to make its length 4 rods less, and its breadth 5 rods less, its contents would be 210 square rods less than its present surface. What are its present length, breadth, and contents?

Ans. Length, 30 rods; breadth, 20 rods; contents, 600 square rods.

17. A person exchanged 12 bushels of wheat for 8 bushels of barley, and £2 16*s.*; offering, at the same time, to sell a certain quantity of wheat for an equal quantity of barley, and £3 15*s.* in cash, or for £10 in cash. Required the prices of the wheat and barley per bushel.

Ans. Wheat at 8 shillings, barley at 5 shillings, per bushel.

18. A farmer, having 89 oxen and cows, found, after he had sold 4 oxen and 20 cows, he had 7 more oxen than cows. What number had he of each at first? *Ans.* 40 oxen and 49 cows.

19. A and B driving their turkeys to market, A says to B,

give me 5 of your turkeys, and I shall have as many as you. B replies, but give me 15 of yours, and then yours will be $\frac{3}{7}$ of mine. What number of turkeys had each?

Ans. A 45 and B 55 turkeys.

20. It is required to find two such numbers, that if $\frac{1}{3}$ of the first be added to $\frac{1}{4}$ of the second, the sum shall be 25; but, if $\frac{1}{6}$ of the second be taken from $\frac{1}{4}$ of the first, the remainder will be 6. *Ans.* 48 and 36.

21. What fraction is that, if 5 be added to its numerator, its value is 2, but if 2 be added to its denominator, its value is $\frac{1}{2}$?

Ans. $\frac{3}{4}$.

22. B says to C, if 3 years were taken from your age and added to mine, I should be twice as old as you. C replies, if 3 years were taken from your age and added to mine, our ages would be the same. Required their ages.

Ans. B's age 21, C's age 15 years.

23. It is required to find two numbers, so that $\frac{2}{3}$ of the first added to $\frac{3}{4}$ of the second shall be $15\frac{2}{3}$, and if $\frac{1}{7}$ of the second be subtracted from $\frac{3}{4}$ of the first, the remainder shall be $5\frac{11}{14}$.

Ans. 10 and 12.

24. It is required to divide 50 into two such parts that $\frac{3}{8}$ of the larger shall be equal to $\frac{2}{3}$ of the smaller.

Ans. 32 and 18.

25. A gentleman, at the time of his marriage, found that his wife's age was to his as 3 to 4; but, after they had been married 12 years, her age was to his as 5 to 6. Required their ages at the time of their marriage.

Ans. The man's age 24, his wife's 18 years.

26. A farmer hired a laborer for ten days, and he agreed to pay him 12 shillings for every day he labored, and he was to forfeit 8 shillings for every day he was absent, and he received at the end of his time 40 shillings. How many days did he labor, and how many days was he absent?

Ans. He labored 6 days, and was absent 4.

27. A gentleman bought a horse and chaise for $208, and $\frac{4}{7}$ of the cost of the chaise was equal to $\frac{2}{3}$ the price of the horse. What was the price of each?

Ans. Chaise, $112; horse, $96.

28. A and B engaged in trade, A with $240, and B with $96. A lost twice as much as B; and, upon settling their accounts, it appeared that A had three times as much remaining as B. How much did each lose?

Ans. A lost $96, and B lost $48.

29. Two men, A and B, agree to dig a well in 10 days, but, having labored together 4 days, B agreed to finish the job, which he did in 16 days. How long would it have required A to complete the labor? *Ans.* 9$\frac{3}{5}$ days.

30. A merchant has two kinds of grain, one at 60 cents per bushel, and the other at 90 cents per bushel, of which he wishes to make a mixture of 40 bushels that may be worth 80 cents per bushel. How many bushels of each must he use?

Ans. 13$\frac{1}{3}$ bushels of 60 cents, 26$\frac{2}{3}$ of 90 cents.

31. A farmer has 30 bushels of oats, at 30 cents per bushel, and which he would mix with corn at 70 cents per bushel, and barley at 90 cents per bushel, so that the whole mixture may consist of 200 bushels, at 80 cents per bushel. How many bushels of corn, and how many of barley, must he mix with the oats? *Ans.* 10 bushels of corn, and 160 of barley.

32. A drover sold 6 of his oxen and 8 of his cows, and he then found he had twice as many oxen as cows. But after he had sold 10 more of his oxen, he found he had 2 more oxen than cows. How many had he of each at first?

Ans. 30 oxen and 20 cows.

33. Four times the larger of two numbers is equal to six times the less, and their sum is 15. Required the numbers.

Ans. 9 and 6.

34. A and B can perform a piece of work in 6 days, A and C in 8 days, and B and C in 12 days. In what time would each

of them perform the work alone, and how long would it take them to perform the work together?

Ans. A would do the work in $9\frac{3}{5}$ days, B in 16 days, C in 48 days, A, B and C together, in $5\frac{1}{3}$ days.

35. A gentleman left a sum of money to be divided among his four sons, so that the share of the oldest was $\frac{1}{2}$ of the shares of the other three, the share of the second $\frac{1}{3}$ of the sum of the other three, and the share of the third $\frac{1}{4}$ of the sum of the other three; and it was found that the share of the oldest exceeded that of the youngest by $14. What was the whole sum; and what was the share of each person?

Ans. Whole sum, $120; oldest son's share, $40; second son's, $30; third son's, $24; youngest son's, $26.

SECTION XI.

NEGATIVE QUANTITIES.

Art. 153. The student will sometimes find that, on account of his misconception of the question, he has added a quantity which should have been subtracted, or that he has subtracted a quantity which should have been added.

This may be illustrated by the following

EXAMPLES.

1. The length of a certain field is a, and its breadth is b; how much must be added to its breadth that its contents may be m?

Let $x =$ the quantity to be added to its breadth.

Then $b+x=$ the breadth.

And $a(b+x)=m$, the contents.

$$ab+ax=m.$$

$$b+x=\frac{m}{a}.$$

$$x=\frac{m}{a}-b.$$

2. Let the length of the field be 10 rods, and its breadth 6 rods; how many rods must be added to its breadth, that the contents of the field may be 80 square rods?

Let $x =$ the quantity to be added to the breadth. Then, by the above formula,

$$x=\frac{m}{a}-b=\frac{80}{10}-6=2 \text{ rods, the quantity to be added.}$$

VERIFICATION.

$$10\times\overline{6+2}=80 \text{ square rods.}$$

3. Let the length of the field be 10 rods, the breadth 8 rods; it is required to find what quantity must be added to the breadth that the contents may be 60 square rods.

By the formula,

$$x=\frac{m}{a}-b=\frac{60}{10}-8=-2 \text{ rods.}$$

We perceive by the above that it is -2 rods which are to be added, and not $+2$ rods; but we add quantities together in Algebra by simply writing them one after the other, with their respective signs, so that -2 added to $+8$ becomes $8-2=6$, the answer, which is the same as subtracting $+2$ from $+8$. And, in general, adding a minus quantity brings the same result as subtracting a plus quantity of equal value, and *vice versa.*

VERIFICATION.

$$10\times\overline{8-2}=60 \text{ square rods. } \textit{Ans.}$$

4. Suppose the field to be 10 rods long and 8 rods wide, it is required to ascertain how much must be subtracted from its width that its contents may be 60 square rods.

To subtract a minus quantity is the same as to add a plus quantity. If, therefore, we change the sign of x in the formula first obtained, x will then express how much is to be subtracted.

Thus, $$-x=\frac{m}{a}-b,$$

or, $$x=b-\frac{m}{a}=8-\frac{60}{10}=2 \text{ rods.}$$

VERIFICATION.

$$10\times\overline{8-2}=60 \text{ square rods.}$$

5. If the field were 10 rods long and 8 rods wide, how many rods must be taken from its width that its contents may be 100 square rods?

By the formula,

$$x=b-\frac{m}{a}=8-\frac{100}{10}=-2 \text{ rods.}$$

That is, -2 is to be subtracted from $+8$; or, as we perform subtraction in Algebra by changing the sign of the subtrahend, and thus annexing it to the minuend, we have

$$8-(-2)=8+2=10;$$

so that, in general, subtracting a minus quantity is the same as adding a plus quantity of equal value.

6. John Smith, at the time of his marriage, was 50 years old, and his wife was 40. When will his age be twice that of his wife?

Let x=the time.

Then, $50+x=2\times\overline{40+x}$.

$50+x=80+2x$.

And, $x=50-80=-30$ years.

As the answer is -30 years, it is evident that he is not *now* twice as old as his wife, but 30 years ago his age was twice hers.

VERIFICATION.

$$50-30=\overline{40-30}\times 2.$$
$$20=20.$$

7. J. Jones is 40 years old, his wife 30. When will they both be of the same age?

Let x=the time.

Then, $40+x=30+x$.

$40-30=x-x$.

And, $10=0$.

The answer being zero, it is certain they never will be of the same age, but that one will always be 10 years older than the other.

8. What fraction is such, that if 2 be added to its numerator its value is $\frac{1}{4}$, or if 2 be added to its denominator its value is $\frac{1}{2}$?

Ans. $\frac{-5}{-12}$.

9. What fraction is such, that if 7 be added to the numerator its value is nothing, but if 2 be added to its denominator its value is infinite?

Ans. $\frac{-7}{-2}$.

10. What fraction is such, that, if 4 be added to its numerator its value is nothing, but if 10 be subtracted from its denominator its value is 1?

THE COURIERS.

1. Two couriers set out at the same time from A and C, and travel towards each other until they meet. The distance from A to C is m miles. The first courier travels a miles per hour, and the second b miles per hour. How far from A and C will they meet?

Let us suppose them to meet at B.

And let $x =$ the distance A B.

And $y =$ the distance B C.

Then $x+y =$ A C $= m$.

As the first travels x miles at the rate of a miles per hour, to find the time he will travel this distance, we say,

As a miles : x miles : : 1 hour : $\frac{x}{a} =$ the time the first courier will travel the distance A B.

And, as b miles : y miles : : 1 hour : $\frac{y}{b}$ hours $=$ the time the second courier will travel the distance B C.

As both couriers set out at the same time, and arrive at the same time at C,

Therefore $$\frac{x}{a}=\frac{y}{b}.$$

And $$x=\frac{ay}{b}.$$

If we substitute this value of x in the first equation, we have

$$\frac{ay}{b}+y=m.$$

And $$ay+by=bm.$$

Hence $$y=\frac{bm}{a+b}.$$

Substituting this value of y in the equation $x=\frac{ay}{b}$, we have

$$x=\frac{a}{b}\times\frac{bm}{a+b}=\frac{abm}{ab+b^2}=\frac{am}{a+b}.$$

The values of x and y in the above equation are both positive. Therefore, whatever value we may assign to a, b and m, it will answer the conditions of the question.

This may be illustrated by the following question:

2. Two men, A and B, set out from two places, distant from each other 144 miles, and travel towards each other. A goes 12 miles an hour, and B four miles an hour. How far must each travel before they meet?

By the above formulæ,

$$x=\frac{am}{a+b}=\frac{12\times144}{12+4}=108 \text{ miles, the distance A travels.}$$

And $$y=\frac{bm}{a+b}=\frac{4\times144}{12+4}=36 \text{ miles, the distance B travels.}$$

VERIFICATION.

$$108+36=144 \text{ miles.}$$

3. If the couriers were to set out at the same time from A and B, and travel towards C, both going the same direction, the first going a miles per hour, and the second b miles per hour, and the distance A B being m, how far would each travel before they meet, suppose at a point C?

F A B C D

Let x = the distance A C.
And y = the distance B C.
Then $x-y=\text{AC}-\text{BC}=\text{AB}=m.$

By performing the same operation as in the first question, we find

$$\frac{x}{a}=\frac{y}{b},$$

$$\text{and } x=\frac{ay}{b}.$$

Therefore $\frac{ay}{b}-y=m.$

And $ay-by=bm.$

Whence $y=\frac{bm}{a-b}.$

Substitute this last value of y in the former equation, and we have

$$x=\frac{ay}{b}=\frac{a}{b}\times\frac{bm}{a-b}=\frac{abm}{ab-b^2}=\frac{am}{a-b}.$$

Here it is evident that the values of x and y will not be positive, unless a be greater than b; or, in other words, unless the courier which sets out from A travels faster than the one that sets out from B, he will never overtake him.

4. Suppose the first courier to travel 9 miles per hour, and the second 6 miles per hour, and the distance A B to be 18 miles, and it was required to find how far each would travel before the one overtook the other.

Then $a=9$, $b=6$, and $m=18$.

And, by the first formula,

$x=\frac{am}{a-b}=\frac{9\times18}{9-6}=54$ miles, the distance the first courier would travel.

And, by the second formula,

$y=\frac{bm}{a-b}=\frac{6\times18}{9-6}=36$ miles, the distance the second courier would travel.

We perceive, by the above operation, that the point C, where the couriers meet, is 54—36=18 miles further from A than B is, which is equal to the distance m.

5. Again, let $a=6$, $b=9$, and $m=18$; or, suppose the first courier sets out from A and travels 6 miles an hour, and the second sets out at the same time from B and travels in the same direction towards C at the rate of 9 miles per hour. What distance will each travel before they meet?

By the first formula,

$$x=\frac{am}{a-b}=\frac{6\times18}{6-9}=-36 \text{ miles, the first travels.}$$

By the second formula,

$$x=\frac{bm}{a-b}=\frac{9\times18}{6-9}=-54 \text{ miles, the second travels.}$$

Here the values of x and y are both negative. Now, how shall we interpret this result? What is the meaning of the negative sign, in this case?

To understand this, we must observe that we began by supposing the parties to be travelling towards C, and any motion in this direction would have been indicated in this example, as it has been in the preceding examples, by the sign $+$. But, when the sign $+$ is taken to indicate motion in one direction, the opposite sign — must indicate motion in the opposite direction. Hence the minus sign, resulting as above, indicates that the parties, in order to meet, must travel, not towards C, as we at first supposed, but in the opposite direction, towards F, a point 36 miles from A, and 54 miles from B, where they will meet.

6. Again, let $a=6$, $b=6$, and $m=18$; or, we will suppose the couriers both to start at the same time from A and B, and both to travel in the same direction towards C, and both travelling at the same rate of 6 miles per hour, the distance A B being 18 miles. What distance will each travel before they meet?

By the first formula, $x=\frac{am}{a-b}$, or $\frac{am}{a-a}=\frac{am}{0}$,

or $$x=\frac{6\times 18}{6-6}=\frac{108}{0}.$$

By the second formula, $y=\frac{bm}{a-b}$, or $\frac{bm}{a-a}=\frac{bm}{0}$,

or $$y=\frac{6\times 18}{6-6}=\frac{108}{0}.$$

As both couriers are travelling in the same direction, and at the same rate, it is certain they will never meet, but the distance between them will continue the same.

154. Therefore, the expression $\frac{am}{0}$ or $\frac{108}{0}$, or any quantity with zero for a denominator, is the symbol for infinity; for it is well known that the value of a fraction depends on the number of times the numerator contains the denominator, or the number of times the denominator may be taken from the numerator, until nothing shall remain.

It is certain that, if a be greater than b, however small the difference, the couriers will eventually meet; but, if the difference between a and b be less than any assignable quantity, then x and y may be considered *infinite.*

Again, let $a=b$, and $m=0$.

Then $$x=\frac{am}{a-b}=\frac{a\times 0}{0}=\frac{0}{0}.$$

And $$y=\frac{bm}{a-b}=\frac{b\times 0}{0}=\frac{0}{0}.$$

From the above we infer that x and y are equal, and that each is equal to the other.

Thus, $$x=x.$$

This is an *identical equation,* and the values of the unknown quantities cannot be known by it.

And, as $m=0$, it is evident, that as both couriers start from the same point, and travel at the same rate, and in the same direction, they will always be together, and therefore cannot meet.

We say, therefore, that the $\frac{0}{0}$, in this case, is an expression of an Indeterminate Quantity, because that x and y may be any quantities whatever.

But it is not true that the expression $\frac{0}{0}$ is always the sign of an indeterminate quantity.

155. In fractions, when the numerator and denominator have a common factor, and which in some cases becomes zero, and makes the fraction assume the form of $\frac{0}{0}$, but which, without that factor, has a definite value, the expression is not indeterminate.

The following fractions are examples of this kind:

$$\frac{m(m^2-n^2)}{n(m-n)}.$$

Now, if $m=n$, the value of the quantity is $\frac{0}{0}$.

But, on examination, we perceive that both the numerator and denominator have the common factor $m-n$.

Therefore, by dividing both terms of the expression by $m-n$, it becomes $\frac{m(m+n)}{n}$, which, if $m=n$, is equal to $2m$.

The value of the expression $\qquad \frac{x-1}{x-1}$,

if we divide both terms by $x-1$, is 1; but, if $x=1$, the value is $\frac{0}{0}$.

Again, let $\qquad x=\frac{m^3-n^3}{m-n}.$

Then, if $m=n$, the value of $x=\frac{0}{0}$.

But, if we divide both terms by the common factor $m-n$, its value is m^2+mn+n^2, and then, on the supposition that $m=n$, its value will be $3m^2$.

INDETERMINATION.

156. In investigating the theory of indetermination, we find many curious results and apparent absurdities.

This will appear evident by investigating the following problems.

1. If it be admitted that $a=1$ and $x=1$, it may be shown that 1 is 2 and 2 is nothing, or any assignable quantity.

Let $a=x$.

Multiplying both terms of the equation by x, we have

$$ax=x^2.$$

Subtracting a^2 from both members,

$$ax-a^2=x^2-a^2.$$

Resolving both terms into factors,

$$a(x-a)=(x-a)(x+a).$$

Dividing by $x-a$,

$$a=x+a.$$

Substituting a for its value x,

$$a=a+a.$$

Dividing both terms by a,

$$1=1+1=2.$$

Again, we have found above that

$$x^2-a^2=ax-a^2.$$

Dividing both terms by the common factor $x-a$, we have

$$x+a=\frac{ax-a^2}{x-a}.$$

Now, as x and a by the supposition are each equal to 1, we see that

$$1+1=\frac{1\times1-1^2}{1-1}.$$

And $2=\frac{0}{0}$.

Thus it *appears* that we have *clearly* proved that 1 is 2, and 2 any assignable quantity. Q. E. D.

The fallacy is this, that if nothing be divided by nothing the quotient is any assignable quantity.

This principle may be further illustrated by considering the following identical equation.

Let $16=16$.
Resolving into terms, $12+4=12+4$.
Transposing, $4-4=12-12$.
Resolving second term into factors, $4-4=3(4-4)$.
Dividing by $4-4$, $1=3$.

Thus it appears that 1 is 3; and, in the same manner, a unit may be proved to be any definite number.

From various articles in the foregoing section, we infer the following:

1. If zero be multiplied by zero, or any assignable quantity, the product will be zero.

2. If zero be divided by zero, the quotient may be zero, or any assignable quantity.

3. If zero be divided by any quantity, the quotient will be zero.

4. If any quantity be divided by zero, the quotient will be infinity.

5. If any quantity be added to or taken from infinity, the result will be infinity.

6. If zero be multiplied by infinity, the product may be any quantity.

7. If infinity be divided by infinity, the quotient may be any assignable quantity.

8. One infinity may be infinitely larger than another.

SECTION XII.

THEOREM I.

ART. **157.** If a binomial be multiplied into itself, the product will be equal to the sum of the squares of both terms, plus twice the product of the terms.

NOTE. — The theorems in the following section may be illustrated by diagrams, and it would be well for the pupils to draw them.

When a number or quantity is multiplied into itself, the product is a *square*.

EXAMPLES.

1. Multiply $a+b$ into itself.

$$\begin{array}{l}
a+b \\
\underline{a+b} \\
a^2+ab \\
\qquad\underline{ab+b^2} \\
a^2+2ab+b^2
\end{array}
\qquad\qquad
\begin{array}{l}
\text{VERIFICATION.} \\
8+4=12 \\
\underline{8+4=12} \\
64+32 \\
\qquad\underline{32+16} \\
64+64+16=144.
\end{array}
\qquad
\begin{array}{r}
\\
12 \\
\underline{12} \\
144
\end{array}$$

We perceive, by the above operation, that the square of any binomial may be readily obtained.

2. Multiply $3a+2b$ into itself.

$$3a\times 3a+2\times 3a\times 2b+2b\times 2b= \\ 9a^2+12ab+4b^2.$$

3. Multiply $x+2y$ into itself. *Ans.* $x^2+4xy+4y^2$.
4. Multiply $3ab+m$ into itself. *Ans.*
5. Multiply $5y+4x$ into itself. *Ans.*
6. Multiply $2m+3n$ into itself. *Ans.*
7. Multiply $7d+2e$ into itself. *Ans.*
8. Multiply $2n+3w$ into itself. *Ans.*
9. Multiply $5a^2+2b$ into itself. *Ans.*
10. Multiply $1+\frac{1}{4}$ into itself. *Ans.*
11. Multiply $3+\frac{1}{3}$ into itself. *Ans.*
12. Multiply $2+\frac{1}{2}$ into itself. *Ans.*

THEOREM II.

158. If the sum of two numbers or quantities be multiplied by their difference, the product will be equal to the difference of their squares.

Multiply $a+b$ into $a-b$.

$$\begin{array}{l}
a+b \\
\underline{a-b} \\
a^2+ab \\
\qquad\underline{-ab-b^2} \\
a^2 \qquad -b^2
\end{array}
\qquad\qquad
\begin{array}{l}
\text{VERIFICATION.} \\
8+4=12 \\
\underline{8-4=\ 4} \\
64+32 \\
\qquad\underline{-32-16} \\
64 \qquad -16=48.
\end{array}
\qquad
\begin{array}{r}
\\
12 \\
\underline{4} \\
48
\end{array}$$

THEOREM III.

159. If the difference of two numbers or quantities be multiplied into itself, the product will be equal to the sum of their squares, minus twice their product.

EXAMPLES.

1. Multiply $a-b$ into $a-b$.

VERIFICATION.

$$\begin{array}{lll} a-b & 12-3=9 & 9 \\ \underline{a-b} & \underline{12-3=9} & \underline{9} \\ a^2-ab & 144-36 & 81 \\ \quad\underline{-ab+b^2} & \quad\underline{-36+9} & \\ a^2-2ab+b^2. & 144-72+9=81. & \end{array}$$

2. Multiply $3a-2b$ into $3a-2b$. *Ans.* $9a^2-12ab+4b^2$.
3. Multiply $5m-n$ into $5m-n$.
4. Multiply $4ab-x$ into $4ab-x$.
5. Multiply $3a^2-b^3$ into $3a^2-b^3$.
6. Multiply x^4-y^2 into x^4-y^2.

NOTE.—If the square of the difference of two numbers be subtracted from the square of their sum, the remainder will be equal to four times their product.

Thus, $(a+b)^2-(a-b)^2=(a^2+2ab+b^2)-(a^2-2ab+b^2)=4ab$.

THEOREM IV.

160. If twice the product of two quantities be subtracted from the sum of their squares, the remainder will be equal to the square of their difference.

$$(a^2+b^2)-2ab=a^2-2ab+b^2.$$

But this expression, by Problem 3d, is the square of their difference.

VERIFICATION.

Let 9 and 3 be the two numbers.

Then
$$\begin{array}{l} (9^2+3^2)-(2\times9\times3)=(9-3)^2. \\ (81+9)-(54)=36. \\ 90-54=36. \\ \qquad 36=36. \end{array}$$

THEOREM V.

161. If there be two quantities, one of which is divided into any number of parts, the product of the two quantities will be equal to the product of the undivided number into the several parts of the divided number.

Let the two quantities be a and b, and let b be divided into three parts, c, d, and e.

Then $$b=c+d+e.$$

And $$ab=ac+ad+ae.$$

VERIFICATION.

Let the two numbers be 12 and 10, and let 10 be divided into the parts 5, 3, and 2.

Then $$10=5+3+2.$$

And $$12\times10=\overline{12\times5}+\overline{12\times3}+\overline{12\times2}.$$
$$120=60+36+24.$$
$$120=120.$$

THEOREM VI.

162. If any quantity be divided into two parts, the square of this quantity will be equal to the sum of the products of this quantity into its two parts.

Let a represent the quantity, and b and c the parts into which it is divided.

Then $$a=b+c.$$

And $$a\times a=a(b+c).$$
$$a^2=ab+ac.$$

VERIFICATION.

Let 12 be divided into two parts, 9 and 3.

Then $$12=9+3.$$
$$12\times12=12(9+3).$$
$$144=108+36=144.$$

THEOREM VII.

163. If any quantity or number be divided into two parts, the product of the whole and one of the parts will be equal to the product of the two parts, plus the square of the aforesaid part.

Let a represent the whole quantity, and b and c the parts.

Then $a=b+c.$

Multiplying both sides of the equation by b, we have

$$ab=b^2+bc.$$

VERIFICATION.

Let 12 represent the number, and 9 and 3 the parts into which it is divided.

Then $12=9+3.$

Multiplying both parts of the equation by 9, we have

$$9\times12=9(9+3).$$
$$108=81+27=108.$$

THEOREM VIII.

164. If any quantity be divided into two parts, the square of the whole quantity will be equal to the squares of the two parts, plus twice their product.

Let a represent any quantity, and b and c the parts into which it is divided.

Then $a=b+c.$

By squaring both sides of the equation, we have

$$a^2=b^2+2bc+c^2.$$

VERIFICATION.

Let 9 be divided into two parts, 6 and 3.

Then $9=6+3.$

By squaring both parts of the equation, we have

$$9^2=(6+3)^2.$$
$$81=36+36+9=81.$$

THEOREM IX.

165. If any number or quantity be divided into two equal parts, and into two unequal parts, the square of one of the equal parts will be equal to the product of the two unequal parts, plus the square of half the difference of the two unequal parts.

Let a represent one of the equal parts, and b and c the two unequal parts.

Then $$a=\frac{b+c}{2}.$$

And $$2a=b+c.$$
$$4a^2=b^2+2bc+c^2.$$

We now add $-4bc$ to both sides of the equation.

And $$4a^2-4bc=-4bc+b^2+2bc+c^2.$$
$$4a^2-4bc=b^2-2bc+c^2.$$
$$a^2-bc=\frac{b^2-2bc+c^2}{4}.$$
$$a^2=bc+\frac{b^2-2bc+c^2}{4}.$$

VERIFICATION.

Let 12 be divided into two equal parts, 6 and 6; and into two unequal parts, 9 and 3.

Then $$6^2=9\times3+\frac{9^2-2\times9\times3+3^2}{4}.$$

And $$36=27+\frac{81-54+9}{4}.$$
$$36=27+9=36.$$

THEOREM X.

166. If any quantity, $2a$, be divided into two equal parts, and if any quantity, b, be added to $2a$, the product of $2a+b$ into b, plus the square of a, will be equal to the square of $a+b$. Then, by the proposition, $2a+b$ will be the whole quantity.

Multiplying by b, we have
$$b(2a+b)=2ab+b^2.$$

By adding a^2 to each member of the equation, we have
$$b(2a+b)+a^2=a^2+2ab+b^2.$$

Therefore $$b(2a+b)+a^2=(a+b)^2.$$

VERIFICATION.

Let $a=10$, and $b=2$.

Then $2(\overline{2\times10}+2)+10^2=(10+2)^2$;

$$144 = 144$$

THEOREM XI.

167. If any quantity be divided into two parts, the square of this quantity, and one of the parts, will be equal to twice the product of the whole quantity and that part, plus the square of the other part.

Let the whole quantity be denoted by a, and the parts by b and c.

Then $a=b+c$.

And $a-c=b$.

$$a^2-2ac+c^2=b^2.$$

$$a^2+c^2=2ac+b^2.$$

VERIFICATION.

Let $12=3+9$.

Then $12^2+9^2=\overline{2\times12\times9}+3^2$.

And $144+81=216+9$.

$$225=225.$$

THEOREM XII.

168. If any quantity be divided into any two parts, four times the product of the whole quantity into one of the parts, plus the square of the other part, will be equal to the square of the quantity which consists of the whole and the first-mentioned part.

Let a represent the quantity, and b and c the two parts into which it is divided.

Then $a=b+c$.

Multiplying both members of the equation by $4b$, we shall have $4b\times a=4b\times(b+c)$.

$$4ab=4b^2+4bc.$$

We now add c^2 to both members.

$$4ab+c^2=c^2+4bc+4b^2.$$

Or $$4ab+c^2=(c+2b)^2.$$

VERIFICATION.

Let $a=12$, and $b=9$, and $c=3$.

Then $12=9+3$.

And $\overline{4\times\overline{12\times9}}+3^2=(3+\overline{2\times9})^2.$

$$441 = 441$$

Theorem XIII.

169. If any quantity be divided into two equal parts, and also into two unequal parts, the sum of the squares of the two unequal parts will be double the square of half the quantity, plus twice the square of the quantity which consists of the difference between half the quantity and the larger of the unequal parts of the quantity.

Let $2a$ represent the quantity, and a = one of the equal parts, and b = half the difference between the equal and unequal parts.

Then $a+b$ = the larger part.

And $a-b$ = the less part.

And $(a+b)^2+(a-b)^2$ = the sum of their squares.

But $(a^2+2ab+b^2)+(a^2-2ab+b^2)=2a^2+2b^2$.

And $2a^2+2b^2$ = twice the square of half the quantity, plus twice the square of half the difference between the equal and unequal parts; that is, the difference between half the quantity and the larger of the unequal parts.

VERIFICATION.

Let $10=7+3$; $10\div2=5$; $7-5=2$.

Then $(5+2)^2+(5-2)^2=\overline{2\times5^2}+\overline{2\times2^2}.$

And $49+9=50+8.$

$$58=58.$$

THEOREM XIV.

170. If any quantity, $2a$, be divided into two equal quantities, a and a; and, if any quantity, b, be added to $2a$, the square of $2a+b$, plus the square of b, will be equal to twice the square of a, plus twice the square of $a+b$.

Now $(2a+b)^2+b^2=4a^2+4ab+b^2+b^2=4a^2+4ab+2b^2$.

But $4a^2+4ab+2b^2=2a^2+2(a+b)^2$.

Therefore $(2a+b)^2+b^2=2a^2+2(a+b)^2$.

VERIFICATION.

Let $a=10$; and $b=4$.

Then $(2\times 10+4)^2+4^2=2(10^2)+2(10+4)^2$.

And $576+16=200+392$.

Therefore $592=592$.

SECTION XIII.

INVOLUTION.

ART. 171. Involution is the raising of powers from any proposed root; or, the method of finding the square, cube, biquadrate, &c., of any given quantity.

172. A *power* is the product of any quantity multiplied into itself a certain number of times, and the degree of the power is denoted by an exponent written over the root. Thus a^3 is the third power of a, and a is the root.

173. The *exponent*, or *index*, shows how many times the root has been used as a factor.

Thus, $a\times a\times a\times a=a^4$, and $x\times x=x^2$.

174. When a quantity is written without any index, its index is uniformly considered a unit. Thus, $a=a^1$, and $x=x^1$. Therefore, to raise any quantity to any required power, the pupil will see the propriety of the following

Rule. *Multiply the index of the quantity by the index of the power to which it is to be raised, and the result will be the power required.*

Or, multiply the quantity into itself as many times, less one, *as is denoted by the index of the power, and the last product will be the answer.*

175. When the sign of any simple quantity is +, all the powers of it will be +; and when the sign is —, all the *even* powers will be +, and the *odd* powers —, as is evident from multiplication.

EXAMPLES.

1. What is the fifth power of a? *Ans.* a^5.
2. What is the third power of ax? *Ans.* a^3x^3.
3. Required the square of a^2x. *Ans.* a^4x^2.
4. Required the cube of $-3a^2$. *Ans.* $-27a^6$.
5. Required the fourth power of $-ab^2c^3$. *Ans.* $a^4b^8c^{12}$.
6. Required the square of $-\frac{2ax^2}{3b}$. *Ans.* $\frac{4a^2x^4}{9b^2}$.
7. Required the fifth power of $2ab^2x^3$. *Ans.* $32a^5b^{10}x^{15}$.
8. Required the sixth power of $\frac{2}{3}a^3x^2$. *Ans.* $\frac{64}{729}a^{18}x^{12}$.
9. Required the third power of $2a^{-2}$. *Ans.* $8a^{-6}$.
10. Required the fourth power of $-3m^{-3}$. *Ans.* $81m^{-12}$.
11. Required the *m*th power of a^n. *Ans.* a^{mn}.
12. Required the fourth power of $2x^m$. *Ans.* $16x^{4m}$.
13. Required the third power of $\frac{3a^2b^3}{4xy^4}$. *Ans.* $\frac{27a^6b^9}{64x^3y^{12}}$.

176. Polynomials are involved by multiplying the quantity by itself as many times, *wanting one*, as there are units in the exponent of the power.

14. Let $a+b$ be raised to the fifth power.

$(a+b)^1=a+b$ 1st power.

$a+b$

a^2+ab

$+ab+b^2$

$(a+b)^2=a^2+2ab+b^2$ 2d power.

$a+b$

$a^3+2a^2b+ab^2$

$+a^2b+2ab^2+b^3$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$ 3d power.

$a+b$

$a^4+3a^3b+3a^2b^2+ab^3$

$+a^3b+3a^2b^2+3ab^3+b^4$

$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$ 4th power.

$a+b$

$a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4$

$+a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5$

$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$ 5th power.

Required the third power of $a-b$.

$(a-b)^1=a-b$ 1st power.

$a-b$

a^2-ab

$-ab+b^2$

$(a-b)^2=a^2-2ab+b^2$ 2d power.

$a-b$

$a^3-2a^2b+ab^2$

$-a^2b+2ab^2-b^3$

$(a-b)^3=a^3-3a^2b+3ab^2-b^3$ 3d power.

15. Required the fifth power of $x-2y$.

Ans. $x^5-10x^4y+40x^3y^2-80x^2y^3+80xy^4-32y^5$.

16. Required the third power of $a-b+1$.

Ans. $a^3-3a^2b+3a^2+3ab^2-6ab+3a-b^3+3b^2-3b+1$.

17. Required the second power of $2x^2-3x+4$.

Ans. $4x^4-12x^3+25x^2-24x+16$.

18. Required the sixth power of $x-2$.

Ans. $x^6-12x^5+60x^4-160x^3+240x^2-192x+64$.

19. Required the second power of $\frac{2x^2y^{-4}}{3b-4d}$.

Ans. $\frac{4x^4y^{-8}}{9b^2-24bd+16d^2}$.

20. Required the fourth power of a^m-a^n.

Ans. $a^{4m}-4a^{3m+n}+6a^{2m+2n}-4a^{m+3n}+a^{4n}$.

21. What is the second power of $2x^2-3x+\frac{1}{2}$?

Ans. $4x^4-12x^3+11x^2-3x+\frac{1}{4}$.

22. What is the third power of $a+2b-c$?

Ans. $a^3+6a^2b-3a^2c+12ab^2-12abc+3ac^2+8b^3-12b^2c+6bc^2$
$-c^3$.

23. What is the fourth power of $a+b+c+d$?

Ans. $a^4+4a^3b+6a^2b^2+4ab^3+b^4+4a^3c+12a^2bc+12ab^2c+4b^3c$
$+4a^3d+12a^2bd+12ab^2d+4b^3d+6a^2c^2+12abc^2+6b^2c^2+12a^2cd$
$+24abcd+12b^2cd+6a^2d^2+12abd^2+6b^2d^2+4ac^3+12ac^2d+$
$12acd^2+4ad^3+4bc^3+12bc^2d+12bcd^2+4bd^3+c^4+4c^3d+6c^2d^2+$
$4cd^3+d^4$.

24. What is the second power of x^3+2x^2+x+2?

Ans. $x^6+4x^5+6x^4+8x^3+9x^2+4x+4$.

25. What is the second power of $\frac{a}{b}-\frac{b}{a}$? *Ans.* $\frac{a^2}{b^2}-2+\frac{b^2}{a^2}$.

26. What is the third power of x^2-x-1?

Ans. $x^6-3x^5+5x^3-3x-1$.

27. What is the third power of $a-b-2c^2-d^3$?

Ans. $a^3-3a^2b+3ab^2-b^3-6a^2c^2+12abc^2-6b^2c^2-3a^2d^3+6abd^3$
$-3b^2d^3+12ac^4+12ac^2d^3+3ad^6-12bc^4-12bc^2d^3-3bd^6-8c^6-$
$12c^4d^3-6c^2d^6-d^9$.

SECTION XIV.

EVOLUTION, OR THE EXTRACTION OF ROOTS.

ART. 177. Evolution is the reverse of involution, being the method of finding the roots of any given quantity. It will, therefore, be necessary to trace back the steps of the operation in involution.

Hence, to find any root of a monomial, we adopt the following

RULE. *Extract the required root of the coefficient for the coefficient of the answer, and the root of the quantity subjoined for the literal part of the answer.*

178. If the quantity proposed be a fraction, its root will be found by taking the root both of its numerator and denominator.

179. The square root, the fourth root, or any other even root of an affirmative quantity, may be either plus or minus.

Thus, $\sqrt{a^2}=+a$ or $-a$; and $\sqrt[4]{b^4}=+b$, or $-b$. But the cube root, or any other odd root of a quantity, will have the same sign as the quantity itself. Thus $\sqrt[3]{a^3}=a$; $\sqrt[3]{-a^3}=-a$, and $\sqrt[5]{-a^5}=-a$.

The reason why $+a$ and $-a$ are each the square root of a^2, is obvious; since, by the rule of multiplication, $(+a)\times(+a)$ and $(-a)\times(-a)$ are each equal to a^2.

180. In the case of the cube root, fifth root, &c., of a negative quantity, the rule is equally plain; since, by multiplying, we have $(-a)\times(-a)\times(-a)=-a^3$.

It may also be stated here that any even root of a negative uantity is unassignable; or, as it is usually called, imaginary.

Thus, $\sqrt{-a^2}$ cannot be determined, as there is no quantity, either positive or negative, that, when multiplied by itself, will produce $-a^2$.

EXAMPLES.

1. Find the square root of $9a^2$.

Here $\sqrt{9a^2}=\sqrt{9}\times\sqrt{a^2}=3\times a=3a$. *Ans.*

2. What is the cube root of $8x^3$?

Here $\sqrt[3]{8x^3}=\sqrt[3]{8}\times\sqrt[3]{x^3}=2\times x=2x$. *Ans.*

3. It is required to find the square root of $\frac{a^2b^2}{c^4}$.

$$\text{Here}\quad \sqrt{\frac{a^2b^2}{c^4}}=\frac{\sqrt{a^2b^2}}{\sqrt{c^4}}=\frac{ab}{c^2}.\quad \textit{Ans.}$$

4. What is the cube root of $-\frac{8a^3b^6}{27c^3}$?

$$\text{Here}\ -\sqrt[3]{\frac{8a^3b^6}{27c^3}}=-\frac{\sqrt[3]{8}\times\sqrt[3]{a^3b^6}}{\sqrt[3]{27}\times\sqrt[3]{c^3}}=-\frac{2\times ab^2}{3\times c}=-\frac{2ab^2}{3c}.\quad \textit{Ans.}$$

5. What is the square root of $16a^4b^8$? *Ans.* $4a^2b^4$.

6. What is the cube root of $-125x^3y^6$? *Ans.* $-5xy^2$.

7. What is the fourth root of $81a^4b^8$? *Ans.* $3ab^2$.

8. What is the fifth root of $\frac{32m^5n^{10}}{243}$? *Ans.* $\frac{2mn^2}{3}$.

9. What is the sixth root of $\frac{729a^6b^{12}}{4096}$? *Ans.* $\frac{3ab^2}{4}$.

NOTE.—Fractions should first be reduced to their lowest terms.

10. Required the square root of $\frac{200a^7}{512a^5}$. *Ans.* $\frac{5a}{8}$.

EVOLUTION OF POLYNOMIALS.

181. To extract the square root.

Since the square of $a+b$ is $a^2+2ab+b^2$, in order to obtain the square root of $a^2+2ab+b^2$, we must consider by what process the quantity $a+b$ can be generally derived from it.

Now, in the first place, we observe that a, the first term of the root, is the square root of a^2, the first term of the square and, in addition to this, there still remains $2ab+b^2$, from which b is to be obtained; but $2ab+b^2$ is the same as $(2a+b)b$; and, therefore, b will be determined by dividing the first term of the remainder by *twice* the first term of the root. To complete the operation, *twice* this first term, together with the second, must be multiplied by the second; and, after subtraction, there is no remainder.

182. If the proposed quantity consists of more terms, it is evident that we have only to consider $a+b$ in the place of a, and then, by the same process, another term of the root will be obtained, and so on; and hence we have the following

GENERAL RULE. *Arrange the terms in the order of the magnitudes of the indices of some one quantity.*

Find the square root of the first term, and subtract its square from the proposed quantity.

Bring down the next two terms, and find the next term of the root by dividing this last quantity by twice the first, and affix it, with the proper sign, to the divisor.

Multiply this result by the second term of the root, and bring down to the remainder as many terms as make the number equal to that of the next completed divisor; and thus continue the process, till the root, or the requisite approximation to it, be obtained.

See NATIONAL ARITHMETIC, page 243.

EXAMPLES.

1. Find the square root of $x^6-6x^3y^2+9y^4$.

$$\begin{array}{r|l} & x^6-6x^3y^2+9y^4(x^3-3y^2 \\ & x^6 \\ \hline 2x^3-3y^2) & -6x^3y^2+9y^4 \\ & -6x^3y^2+9y^4. \\ \hline \end{array}$$

183. If the terms had been arranged in the reverse order, as $9y^4-6x^3y^2+x^6$, the root would have been found by a similar process to be $3y^2-x^3$, which differs in its *sign* from the former.

The reason of this is, that the square root of a quantity may be either positive or negative, agreeably to Art. 179; and in the first case we have one sign, in the second the opposite.

2. Find the square root of $4x^4-4x^3-3x^2+2x+1$.

$$\begin{array}{r}
4x^4-4x^3-3x^2+2x+1(2x^2-x-1. \\
\underline{4x^4\qquad\qquad\qquad\qquad} \\
4x^2-x)-4x^3-3x^2\qquad\quad \\
\underline{-4x^3+\ x^2}\qquad\quad \\
4x^2-2x-1)-4x^2+2x+1 \\
\underline{-4x^2+2x+1}
\end{array}$$

3. Extract the square root of $16(a^4+1)-24a(a^2+1)+41a^2$.

Having arranged the terms according to the dimensions of a, we have

$$\begin{array}{r}
16a^4-24a^3+41a^2-24a+16(4a^2-3a+4. \\
\underline{16a^4\qquad\qquad\qquad\qquad\qquad} \\
8a^2-3a)-24a^3+41a^2\qquad\qquad\qquad \\
\underline{-24a^3+\ 9a^2}\qquad\qquad\qquad \\
8a^2-6a+4)32a^2-24a+16\qquad \\
\underline{32a^2-24a+16}\qquad
\end{array}$$

4. Required the square root of

$4a^3-16a^{\frac{9}{4}}x^{\frac{2}{3}}+16a^{\frac{3}{2}}x^{\frac{4}{3}}+20a^{\frac{3}{2}}y^{\frac{5}{6}}c^{\frac{1}{2}}-40a^{\frac{3}{4}}x^{\frac{2}{3}}y^{\frac{5}{6}}c^{\frac{1}{2}}+25cy^{\frac{5}{3}}$.

$$\begin{array}{r}
4a^3-16a^{\frac{9}{4}}x^{\frac{2}{3}}+16a^{\frac{3}{2}}x^{\frac{4}{3}}+20a^{\frac{3}{2}}y^{\frac{5}{6}}c^{\frac{1}{2}}-40a^{\frac{3}{4}}x^{\frac{2}{3}}y^{\frac{5}{6}}c^{\frac{1}{2}}+25cy^{\frac{5}{3}}. \\
4a^3\qquad\qquad\qquad\qquad\qquad\qquad(2a^{\frac{3}{2}}-4a^{\frac{3}{4}}x^{\frac{2}{3}}+5y^{\frac{5}{6}}c^{\frac{1}{2}}. \\
\hline
4a^{\frac{3}{2}}-4a^{\frac{3}{4}}x^{\frac{2}{3}})-16a^{\frac{9}{4}}x^{\frac{2}{3}}+16a^{\frac{3}{2}}x^{\frac{4}{3}}\qquad\qquad\qquad\qquad\qquad\qquad \\
\underline{-16a^{\frac{9}{4}}x^{\frac{2}{3}}+16a^{\frac{3}{2}}x^{\frac{4}{3}}}\qquad\qquad\qquad\qquad\qquad\qquad \\
4a^{\frac{3}{2}}-8a^{\frac{3}{4}}x^{\frac{2}{3}}+5y^{\frac{5}{6}}c^{\frac{1}{2}})20a^{\frac{3}{2}}y^{\frac{5}{6}}c^{\frac{1}{2}}-40a^{\frac{3}{4}}x^{\frac{2}{3}}y^{\frac{5}{6}}c^{\frac{1}{2}}+25cy^{\frac{5}{3}} \\
\underline{20a^{\frac{3}{2}}y^{\frac{5}{6}}c^{\frac{1}{2}}-40a^{\frac{3}{4}}x^{\frac{2}{3}}y^{\frac{5}{6}}c^{\frac{1}{2}}+25cy^{\frac{5}{3}}.}
\end{array}$$

5. Extract the square root of a^2+x^2.

$$a^2+x^2\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7},\text{ \&c.}\right.$$

$$\underline{a^2}$$

$$2a+\frac{x^2}{2a}\Big)\,x^2$$

$$\underline{x^2+\frac{x^4}{4a^2}}$$

$$2a+\frac{x^2}{a}-\frac{x^4}{8a^3}\Big)-\frac{x^4}{4a^2}$$

$$\underline{-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}}$$

$$2a+\frac{x^2}{a}-\frac{x^4}{4a^3}+\frac{x^6}{16a^5}\Big)\,\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$$

$$\underline{\frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}}$$

$$2a+\frac{x^2}{a}-\frac{x^4}{4a^3}+\frac{x^6}{8a^5}-\frac{5x^8}{128a^7}\Big)-\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}-$$

$$\underline{-\frac{5x^8}{64a^6}-\frac{5x^{10}}{128a^8},\text{ \&c.}}$$

6. What is the square root of $x^4-2x^3+3x^2-2x+1$? *Ans.* x^2-x+1.

7. What is the square root of $x^6-2x^5+x^4+2x^3-2x^2+1$? *Ans.* x^3-x^2+1.

8. What is the square root of $a^4+4a^3b+10a^2b^2+12ab^3+9b^4$? *Ans.* $a^2+2ab+3b^2$.

9. Extract the square root of $a^4-2a^3+2a^2-a+\frac{1}{4}$. *Ans.* $a^2-a+\frac{1}{2}$.

10. What is the square root of $4a^2x^4-12a^3x^3+13a^4x^2-6a^5x+a^6$? *Ans.* $2ax^2-3a^2x+a^3$.

11. What is the square root of $\frac{a^2}{b^2}-\frac{4ab}{3bc}+\frac{4b^2}{9c^2}$?

Ans. $\frac{a}{b}-\frac{2b}{3c}$.

EVOLUTION BY DETACHED COEFFICIENTS.

1. What is the square root of $4x^4-4x^3+13x^2-6x+9$?

$$
\begin{array}{rl}
 & 4-4+13-6+9(2-1+3= \\
 & 4 \qquad\qquad\qquad 2x^2-x+3. \\
\hline
4-1) & -4+13 \\
 & -4+\ 1 \\
\hline
4-2+3) & 12-6+9 \\
 & 12-6+9. \\
\hline
\end{array}
$$

2. What is the square root of $9x^6-24x^4+12x^3+16x^2-16x+4$?

$$
\begin{array}{rl}
 & 9+0-24+12+16-16+4(3+0-4+2= \\
 & 9 \qquad\qquad\qquad\qquad 3x^3+0x^2-4x+2= \\
\hline
6+0-4) & +0-24+12+16 \qquad 3x^3-4x+2. \\
 & -24-\ 0+16 \\
\hline
6+0-8+2) & 12+\ 0-16+4 \\
 & 12+\ 0-16+4. \\
\hline
\end{array}
$$

3. What is the square root of $4x^8-4x^5+12x^4+x^2-6x+9$?

$$
\begin{array}{rl}
 & 4+0+0-4+12+0+1-6+9(2+0+0-1+3= \\
 & 4 \qquad\qquad\qquad\qquad 2x^4+0x^3+0x^2-x+3= \\
\hline
4+0+0-1) & +0+0-4+12+0+1 \qquad 2x^4-x+3. \\
 & +0+0-4-\ 0-0+1 \\
\hline
4+0+0-2+3) & 12+0+0-6+9 \\
 & 12+0+0-6+9 \\
\hline
\end{array}
$$

The pupil will perceive that the 5th power of x in the second question, and the 3d, 6th and 7th power of x in the third question, are wanting; therefore their place in the operation must be supplied by zero.

4. What is the square root of $4a^4-16a^3+24a^2-16a+4$?

Ans. $2a^2-4a+2$.

5. What is the square root of $4x^{10}-12x^{6}-12x^{5}+9x^{2}+18x+9$? *Ans.* $2x^{5}-3x-3$.

6. What is the square root of $16x^{4}+24x^{3}+89x^{2}+60x+100$? *Ans.* $4x^{2}+3x+10$.

7. What is the square root of $9x^{6}-12x^{5}+10x^{4}-28x^{3}+17x^{2}-8x+16$? *Ans.* $3x^{3}-2x^{2}+x-4$.

8. What is the square root of $m^{2}+2m-1-\frac{2}{m}+\frac{1}{m^{2}}$?

Ans. $m+1-\frac{1}{m}$.

EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

184. As numbers are not expressed in the same manner as algebraic quantities, it is evident that the same rule for extracting the square root of algebraic quantities will not apply to extracting the roots of numbers without additional considerations. But, if the foregoing rule be assisted by the "Method of Pointing," it will enable us to extract the square root of numbers.

185. Since the square root of 1 is 1;
the square root of 100 is 10;
the square root of 10000 is 100;
the square root of 1000000 is 1000, &c.,

it is evident that the square root of a number of figures *less* than three must consist of only *one* figure; that of a number more than two figures and less than five, of two figures; that of a number more than four figures and less than seven, of three figures, and so on. Whence it follows, that, if a dot be placed over every *alternate* figure, beginning at the unit's place, the number of such points will be the same as the number of figures in the root.

The same rule may be extended to decimals, by first making the number of decimal places even, and then commencing at the unit's place and pointing towards the right hand over every alternate figure, as before; and the number of such points will be the same as the number of decimal places in the root.

EXAMPLES.

1. Extract the square root of 273529.

ARITHMETICAL FORM.	SYMBOLICAL FORM.
$2\dot{7}3\dot{5}2\dot{9}(523$	$273529(500+20+3$
25	$500^2=250000$
$102)235$	$2\times500+20=1020)23529$
204	20400
$1043)3129$	$2\times(500+20)+3=1043)3129$
$3129.$	3129

The pupil will perceive that both these operations are performed by Art. 182.

2. Extract the square root of 45796. *Ans.* 214.
3. Extract the square root of 106929. *Ans.* 327.
4. Extract the square root of 36372961. *Ans.* 6031.
5. Extract the square root of 22071204. *Ans.* 4698.
6. Extract the square root of 33.1776. *Ans.* 5.76.
7. Extract the square root of .9409. *Ans.* .97.
8. Extract the square root of .0029997529. *Ans.* .05477.
9. Extract the square root of .001234. *Ans.* .035128+.
10. Extract the square root of 32176552.863844. *Ans.* 5672.438.

CUBE ROOT.

186. Investigation of a rule for extracting the Cube Root of a compound algebraical quantity.

Since $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, we must have the cube root of the latter quantity $=a+b$; and our object is to determine how it may be deduced from it.

Now, the first term a of the root is the cube root of a^3, and the first term of the proposed quantity; hence, taking away a^3, we have $3a^2b+3ab^2+b^3$ left to enable us to find b; but $3a^2b+3ab^2+b^3=(3a^2+3ab+b^2)b$. It is, therefore, manifest that b will be obtained by dividing the first term of the remainder by *three times* the square of a; and, to complete the divisor, we must add to $3a^2$ *three* times the product of the two terms, or $3ab$, and also the square of the last, b^2. Thus, the second term being found, the repetition of a similar process will evidently lead to the root, whatever number of terms the expression may contain. Hence the following

RULE. *Arrange the terms according to the powers of some letter, and extract the root of the first term, which must be a cube, or some power of a cube; place this root in the quotient, subtract its cube from the first term, and there will be no remainder.*

Bring down the three next terms for a dividend, and put three times the square of the root just found in the divisor's place, and see how often this is contained in the first term of the dividend, and the quotient is the next term of the root.

Add three times the product of the two terms of the root, plus the square of the last term, to the term already in the divisor's place, and the divisor will be completed.

Multiply the complete divisor by the last term of the root; subtract the product from the dividend, and to the remainder connect the three next terms, and proceed as before.

EXAMPLES.

1. Find the cube root of $a^3+3a^2b+3ab^2+b^3$.

$$\begin{array}{rl}
 & a^3+3a^2b+3ab^2+b^3(a+b. \\
 & a^3 \\ \hline
3a^2+3ab+b^2) & +3a^2b+3ab^2+b^3 \\
 & +3a^2b+3ab^2+b^3 \\ \hline
\end{array}$$

2. Extract the cube root of $x^6-3x^5+5x^3-3x-1$.

$$\begin{array}{rl}
 & x^6-3x^5+5x^3-3x-1(x^2-x-1. \\
 & x^6 \\ \hline
3x^4-3x^3+x^2) & -3x^5+5x^3-3x \\
 & -3x^5+3x^4-\ x^3 \\ \hline
3x^4-6x^3+3x+1) & -3x^4+6x^3-3x-1 \\
 & -3x^4+6x^3-3x-1. \\ \hline
\end{array}$$

The first divisor is found thus:

$$3\times x^2\times x^2+3(x^2-x)+(-x)^2=3x^4-3x^3+x^2.$$

And the second thus:

$$3(x^2-x)^2+3(x^2-x)\ (-1)+(-1)^2=3x^4-6x^3+3x+1.$$

3. Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

$$\begin{array}{rl}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1(x^2-2x+1. \\
 & x^6 \\ \hline
3x^4-6x^3+4x^2) & -6x^5+15x^4-20x^3 \\
 & -6x^5+12x^4-\ 8x^3. \\ \hline
3x^4-12x^3+15x^2-6x+1) & 3x^4-12x^3+15x^2-6x+1 \\
 & 3x^4-12x^3+15x^2-6x+1. \\ \hline
\end{array}$$

4. Extract the cube root of $x^3+9x^2+27x+27$.

Ans. $x+3$.

5. Extract the cube root of $1-6y+12y^2-8y^3$.

Ans. $1-2y$.

6. Extract the cube root of $a^6-6a^5+40a^3-96a-64$.

Ans. a^2-2a-4.

7. Extract the cube root of $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$. *Ans.* $a+b+c$.

BY DETACHED COEFFICIENTS.

1. What is the cube root of $x^6+6x^5-40x^3+96x-64$?

$$\begin{array}{rl}
 & 1+6+0-40+0+96-64(1+2-4 \\
 & 1 \\ \hline
1^2\times 3= & 3)\ \ 6 \\ \hline
(1+2)^3= & 1+6+12+\ 8 \\ \hline
1^2\times 3= & \quad 3)-12-48 \\ \hline
 & 1+6+\ 0-40+0+96-64.
\end{array}$$

Hence, $1+2-4=x^2+2x-4$. *Ans.*

2. What is the cube root of $8x^9-36x^7+54x^5-27x^3$?

$$\begin{array}{rl}
 & 8+0-36+0+54+0-27(2+0-3. \\
 & 8 \\
\hline
2^2\times 3 = & 12)+0-36 \\
\hline
 & 8+0-36+0+54+0-27.
\end{array}$$

Hence, $2+0-3=2x^3+0x^2-3x=2x^3-3x$.

3. What is the fourth root of $x^4+8x^3+24x^2+32x+16$?

Ans. $x+2$.

4. What is the cube root of $x^9-3x^6y+3x^3y^2-y^3$?

Ans. x^3-y.

187. Reasoning analogous to that employed in Art. 185 will show, that, if a point be placed over every *third* figure, beginning at the *unit's* place, the number of points thus placed will be the number of digits in the cube root; and attention to Art. 186 will furnish the following operation:

1. Extract the cube root of 1860867.

$$\begin{array}{rrl}
 & \dot{}\;\;\dot{}\;\;\dot{} & a\;+\;b\;+c \\
 & 1860867 & (100+20+3=123. \\
a^3 = & 1000000 & = \text{first subtrahend.} \\
\hline
3a^2 = & 30000)860867 & = \text{first remainder.} \\
\hline
3a^2b = & 600000 & \\
3ab^2 = & 120000 & \\
b^3 = & 8000 & \\
\hline
 & 728000 & = \text{second subtrahend.} \\
\hline
3(a+b)^2 = & 43200)132867 & = \text{second remainder.} \\
\hline
3(a+b)^2c = & 129600 & \\
3(a+b)c^2 = & 3240 & \\
c^3 = & 27 & \\
\hline
 & 132867 & = \text{third subtrahend.}
\end{array}$$

This process is the origin of the Rule given on page 248 of the Author's NATIONAL ARITHMETIC, to which the pupil is referred.

SYMBOLICAL FORM.

$$a+b+c$$
$$1860867(100+20+3$$
$$(100)^3= \quad 1000000 \qquad [=123.$$
$$3(100)^2+3(100)2+(20)^2=36400)860867$$
$$728000$$
$$3(100+20)^2+3(100+20)3+3^2=44289)132867$$
$$132867.$$

2. What is the cube root of 31255875?	*Ans.* 315.
3. What is the cube root of 37259704?	*Ans.* 334.
4. What is the cube root of 116930169?	*Ans.* 489.
5. What is the cube root of 508.169592?	*Ans.* 7.98.
6. What is the cube root of .724150792?	*Ans.* .898.

188. To extract any root of a compound algebraical quantity.

Since $(a+x)^m=a^m+ma^{m-1}x+$ &c., it is obvious, that when the quantities are properly arranged, and the first term of the root is found, the second term of the mth root will be obtained by dividing the second term of the proposed quantity by ma^{m-1}, or by m times the first term, raised to the $(m-1)$th power.

And, if the root thus found be raised to the mth power, and the result be subtracted from the quantity proposed, and the process be repeated when necessary, any root of a compound quantity may be determined.

The similarity of the processes employed in this and the preceding articles will be immediately noticed, it being observed in the former, the complete powers of a monomial, binomial, trinomial, &c., are subtracted from the proposed quantity by *one*, *two*, *three*, &c., operations; whereas, in the *latter*, the subtraction of the same quantities is effected at *once*. Hence the following

GENERAL RULE. 1. *Arrange the terms so that the highest power shall stand in the first term, and let the next higher occupy the second place.*

2. *Find the root of the first term, and place it in the quotient;*

and, having raised this root to the required power, subtract it from the first term, and then bring down the second term for a dividend.

3. *Involve the root last found to the next inferior power, and multiply it by the index of the given power for a divisor.*

4. *Divide the dividend by the divisor, and the quotient will be the next term of the root.*

5. *Involve the whole root thus found to the required power, which subtract from the given quantity, and divide the first term of the remainder by the same divisor as before.*

6. *Proceed in this manner for the next term of the root, and so proceed until the work is finished.*

See page 255 of the Author's NATIONAL ARITHMETIC.

EXAMPLES.

1. Required the square root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$.

$$a^4-2a^3x+3a^2x^2-2ax^3+x^4(a^2-ax+x^2.$$
$$a^4$$
$$2a^2)-2a^3x$$
$$a^4-2a^3x+a^2x^2$$
$$2a^2)2a^2x^2$$
$$a^4-2a^3x+3a^2x^2-2ax^3+x^4.$$

2. Required the cube root of $x^6+6x^5-40x^3+96x-64$.

$$x^6+6x^5-40x^3+96x-64(x^2+2x-4.$$
$$x^6$$
$$3x^4)6x^5$$
$$x^6+6x^5+12x^4+8x^3$$
$$3x^4)-12x^4$$
$$x^6+6x^5-40x^3+96x-64.$$

3. Required the fourth root of $16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$.

$$\begin{array}{l}
16x^4-96x^3y+216x^2y^2-216xy^3+81y^4(2x-3y. \\
16x^4 \\
\hline
32x^3)-96x^3y \\
\hline
16x^4-96x^3y+216x^2y^2-216xy^3+81y^4.
\end{array}$$

4. Required the cube root of $m^6-6m^5+40m^3-96m-64$.
Ans. m^2-2m-4.

5. Required the fifth root of $32x^5-80x^4+80x^3-40x^2+10x-1$.
Ans. $2x-1$.

SECTION XV.

SURDS, OR RADICAL QUANTITIES.

Art. 189. Surds, or radical quantities, are roots whose values cannot be exactly obtained, being usually expressed by means of the radical sign, or fractional indices; in which latter case the numerator shows the power to which the quantity is to be raised, and the denominator its root.

Thus, $\sqrt{3}$, or $3^{\frac{1}{2}}$, denotes the square root of 3. $\sqrt[3]{a^2}$, or $a^{\frac{2}{3}}$, is the cube root of the square of a; and $a^{\frac{m}{n}}$, or $\sqrt[n]{a^m}$, is the nth root of the mth power of a.

190. The quantity $\sqrt{2}$, or $\sqrt{3}$, is an irrational quantity or surd, because no number, either whole or fractional, can be found, which, when multiplied by itself, will produce either 2 or 3; but their proximate values may be found, to any degree of exactness, by the common rule for extracting the square root.

Problem I.

191. To reduce a rational quantity to the form of a surd, or radical quantity.

Rule. *Raise the quantity to a power corresponding to the index of the surd to which it is to be reduced, and over this new quantity place the radical sign, or proper index, and it will be the form required.*

EXAMPLES.

1. Let 5 be reduced to the form of a square root.

Here $5\times5=5^2=25$; whence $\sqrt{25}$. *Ans.*

2. Reduce $2x^2$ to the form of the cube root.

Here $(2x^2)^3=8x^6$; whence $\sqrt[3]{8x^6}$, or $(8x^6)^{\frac{1}{3}}$, or $8^{\frac{1}{3}}x^{\frac{6}{3}}$. *Ans.*

3. Let $-2x$ be reduced to the form of the cube root.

Here $(-2x)^3=-8x^3$; therefore $\sqrt[3]{-8x^3}$. *Ans.*

4. Let $3a^2$ be reduced to the form of the square root.

Ans. $\sqrt{9a^4}$.

5. Let $\frac{x^2}{3}$ be reduced to the form of the cube root.

Ans. $\sqrt[3]{\frac{x^6}{27}}$.

6. Reduce x^3 to the form of the fifth root. *Ans.* $\sqrt[5]{x^{15}}$.

7. Let $\frac{x^2}{x-y}$ be reduced to the form of the fourth root.

Ans. $\left(\frac{x^6}{(x-y)^4}\right)^{\frac{1}{4}}$.

8. Let $(x-y^2)$ be reduced to the form of the square root.

Ans. $\left((x-y^2)^2\right)^{\frac{1}{2}}$.

192. If a rational quantity be joined to a surd, it may be reduced to the form of a surd by raising the rational part to the required power, and multiplying it by the surd.

9. Let $5\sqrt{7}$ be reduced to a simple radical form.

$5\sqrt{7}=\sqrt{5\times5}\times\sqrt{7}=\sqrt{25}\times\sqrt{7}=\sqrt{175}$. *Ans.*

10. Let $3\sqrt{a}$ be reduced to a simple radical form.

$3\sqrt{a}=\sqrt{3\times3}\times\sqrt{a}=\sqrt{9a}$. *Ans.*

11. Let $3\sqrt[3]{3}$ be reduced to a simple radical form.

$$3\sqrt[3]{3}=\sqrt[3]{3\times3\times3}\times\sqrt[3]{3}=\sqrt[3]{27}\times\sqrt[3]{3}=\sqrt[3]{81}.\quad \textit{Ans.}$$

12. Let $\frac{1}{3}\sqrt{a}$ be reduced to a simple radical form. *Ans.* $\sqrt{\frac{a}{9}}$.

13. Let $\frac{1}{2}\sqrt[4]{b^2}$ be reduced to a simple radical form.

Ans. $\sqrt[4]{\frac{b^2}{16}}$.

14. Let $3\sqrt[n]{m}$ be reduced to a simple radical form.

Ans. $\sqrt[n]{3^n m}$.

15. Let $\frac{x+1}{x-1}\sqrt{\frac{x-1}{x+1}}$ be reduced to a simple radical form.

$$\frac{x+1}{x-1}\sqrt{\frac{x-1}{x+1}}=\sqrt{\left(\frac{x+1}{x-1}\right)^2\left(\frac{x-1}{x+1}\right)}=\sqrt{\frac{x^3+x^2-x-1}{x^3-x^2-x+1}}=$$

$$\sqrt{\frac{x+1}{x-1}},\text{ or }\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}.\quad \textit{Ans.}$$

16. Let $\frac{2x}{3}\sqrt[3]{\frac{9}{4x^2}}$ be reduced to a simple radical form.

Ans. $\sqrt[3]{\frac{2x}{3}}$.

Problem II.

193. To reduce quantities of different indices to others that shall have a given index.

Rule. *Divide the indices of the quantities given by the index under which the quantities are to be reduced, and the quotients will be the new indices for those quantities.*

Then, over the quantities with their new indices place the given index, and they will be the equivalent quantities required.

EXAMPLES.

1. Reduce $4^{\frac{1}{2}}$ and $8^{\frac{1}{3}}$ to other quantities of the same value, each having the common index $\frac{1}{6}$.

Here $\frac{1}{2}\div\frac{1}{6}=\frac{1}{2}\times\frac{6}{1}=\frac{6}{2}=3$, the first index.

And $\frac{1}{3}\div\frac{1}{6}=\frac{1}{3}\times\frac{6}{1}=\frac{6}{3}=2$, the second index.

Whence $(4^3)^{\frac{1}{6}}=4^{\frac{1}{2}}$; and $(8^2)^{\frac{1}{6}}=8^{\frac{1}{3}}$. *Ans.*

194. The truth of this rule will be evident; for if 4 be raised to the 3d power, and the 6th root extracted, that root will be equal to the square root of 4.

Thus, $4\times4\times4=64$; $\sqrt[6]{64}=2$; $\sqrt{4}=2$.

And, if 8 be raised to the 2d power, and the 6th root extracted, the result will be equal to the cube root of 8.

Thus, $8\times8=64$; $\sqrt[6]{64}=2$; $\sqrt[3]{8}=2$.

2. Reduce $3^{\frac{1}{3}}$ and $5^{\frac{1}{2}}$ to the common index $\frac{1}{6}$.
Ans. $\sqrt[6]{3^2}=\sqrt[6]{9}$; $\sqrt[6]{5^3}=\sqrt[6]{125}$.

3. Reduce a^2 and $a^{\frac{1}{2}}$ to quantities that shall have the common index $\frac{1}{4}$. *Ans.* $\sqrt[4]{a^8}$ and $\sqrt[4]{a^2}$.

4. Reduce $3a^{\frac{1}{2}}$ and $2a^{\frac{3}{4}}$ to the quantities that shall have the common index $\frac{1}{8}$. *Ans.* $3\sqrt[8]{a^4}$ and $2\sqrt[8]{a^6}$.

5. Reduce $5x^{\frac{1}{4}}$ and $6y^{\frac{1}{3}}$ to quantities having the common index $\frac{1}{12}$. *Ans.* $5\sqrt[12]{x^3}$ and $6\sqrt[12]{y^4}$.

6. Reduce $a^{\frac{m}{n}}$ and $b^{\frac{p}{s}}$ to quantities having a common index $\frac{1}{ns}$.

$$a^{\frac{m}{n}}=a^{\frac{m}{n}\times\frac{s}{s}}=a^{\frac{ms}{ns}};\text{ and }b^{\frac{p}{s}}=b^{\frac{p}{s}\times\frac{n}{n}}=b^{\frac{np}{ns}}.$$

Therefore $a^{\frac{m}{n}}=(a^{ms})^{\frac{1}{ns}}$; and $b^{\frac{p}{s}}=(b^{np})^{\frac{1}{ns}}$.

PROBLEM III.

195. To reduce surds to a common index.

RULE. *Reduce the indices of the quantities to a common denominator, and then involve each quantity to the power denoted by its numerator.*

EXAMPLES.

1. Reduce $3^{\frac{1}{2}}$ and $4^{\frac{1}{3}}$ to quantities having a common index.

We first reduce the fractional indices, $\frac{1}{2}$ and $\frac{1}{3}$, to a common denominator, and find them to be $\frac{3}{6}$ and $\frac{2}{6}$, which have the same value as $\frac{1}{2}$ and $\frac{1}{3}$.

Hence $3^{\frac{1}{2}}=3^{\frac{3}{6}}=(3^3)^{\frac{1}{6}}=27^{\frac{1}{6}}$ or $\sqrt[6]{27}$.

And $4^{\frac{1}{3}}=4^{\frac{2}{6}}=(4^2)^{\frac{1}{6}}=16^{\frac{1}{6}}$ or $\sqrt[6]{16}$.

2. Reduce $4^{\frac{1}{3}}$ and $6^{\frac{1}{4}}$ to equal quantities, that shall have the same index.

$$\tfrac{1}{3} \text{ and } \tfrac{1}{4} = \tfrac{4}{12} \text{ and } \tfrac{3}{12}.$$

Therefore $4^{\frac{1}{3}}=4^{\frac{4}{12}}=(4^4)^{\frac{1}{12}}=(256)^{\frac{1}{12}}$ or $\sqrt[12]{256}$. *Ans.*

And $6^{\frac{1}{4}}=6^{\frac{3}{12}}=(6^3)^{\frac{1}{12}}=(216)^{\frac{1}{12}}$ or $\sqrt[12]{216}$. *Ans.*

3. Reduce $2^{\frac{2}{3}}$ and $3^{\frac{1}{2}}$ to equal quantities having a common index. *Ans.* $\sqrt[6]{16}$ and $\sqrt[6]{27}$.

4. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{4}}$ to equal quantities having a common index. *Ans.* $\sqrt[4]{a^2}$ and $\sqrt[4]{b}$.

5. Reduce $x^{\frac{1}{m}}$ and $y^{\frac{1}{n}}$ to quantities having a common index. *Ans.* $\sqrt[mn]{x^n}$ and $\sqrt[mn]{y^m}$.

PROBLEM IV.

196. To reduce surds to their most simple form.

RULE. *Resolve the given quantity into two factors, one of which shall be the greatest corresponding power contained in it, and set the root of this power before the remaining factor, with the proper radical sign between them.*

NOTE.—When the given surd contains no factor which is an exact power, it is already in its most simple form. Thus $\sqrt{15}$ cannot be reduced lower, because neither of the factors 5 or 3 is a square.

EXAMPLES.

1. Let $\sqrt{48}$ be reduced to its most simple form.

We divide 48 into two factors, 16 and 3, 16 being the greatest power of the required root. We therefore extract the square root of 16, and write its root, 4, before the other factor, having the sign prefixed to the surd.

Thus $\sqrt{48}=\sqrt{16\times3}=4\sqrt{3}$. *Ans.*

2. Let $\sqrt[3]{108}$ be reduced to its most simple form.

In this question we find the factors of 108 to be 27 and 4, 27 being the largest possible factor of which the cube root could be extracted. The operation, therefore, is

Thus $\sqrt[3]{108}=\sqrt[3]{27\times4}=3\sqrt[3]{4}$. *Ans.*

3. Let $\sqrt{75}$ be reduced to its most simple form.

Ans. $5\sqrt{3}$.

4. Let $\sqrt[4]{80}$ be reduced to its most simple form.

Ans. $2\sqrt[4]{5}$.

5. Reduce $\sqrt{27a^3x^5}$ to its simplest form.

Here $\sqrt{27a^3x^5}=\sqrt{9a^2x^4\times3ax}=\sqrt{9a^2x^4}\times\sqrt{3ax}=3ax^2\sqrt{3ax}$.

6. Reduce $\sqrt[3]{54a^5x^4}$ to its simplest form. *Ans.* $3ax\sqrt[3]{2a^2x}$.

PROBLEM V.

197. When any number or quantity is prefixed to the surd, that quantity must be multiplied by the root of the factor, as in Art. 196, and the product must then be joined to the other part, as before.

EXAMPLES.

1. Let $2\sqrt{32}$ be reduced to its most simple form.

Here $2\sqrt{32}=2\sqrt{16\times2}=2\times4\sqrt{2}=8\sqrt{2}$. *Ans.*

In performing this question we first find the factors of 32, which are 16 and 2.

We then extract the square root of 16, and multiply its root, 4, by the number prefixed to the surd, and find the product to be 8, to which we subjoin the surd 2.

198. This and all similar questions might have been performed by squaring the number prefixed to the surd, and then multiplying this number by the surd. Let this product be divided into two factors, as before, and the square of the former prefixed to the latter will give the answer.

Thus, $2\sqrt{32}=\sqrt{2\times2\times32}=\sqrt{128}=\sqrt{64\times2}=8\sqrt{2}$. *Ans.*

2. Let $5\sqrt[3]{24}$ be reduced to its most simple form.

Here $5\sqrt[3]{24}=5\sqrt[3]{8\times3}=5\times2\sqrt[3]{3}=10\sqrt[3]{3}.$

Or $5\sqrt[3]{24}=\sqrt[3]{5\times5\times5\times24}=\sqrt[3]{3000}=\sqrt[3]{1000\times3}=10\sqrt[3]{3}.$

3. Reduce $2\sqrt[3]{40}$ to simple terms. *Ans.* $4\sqrt[3]{5}.$

PROBLEM VI.

199. A fractional surd may be reduced to a more convenient form by multiplying both the numerator and denominator by such a number or quantity as will make the denominator a complete power of the kind required, and then proceeding as before. [Art. 198.]

EXAMPLES.

1. Let $\sqrt{\frac{2}{5}}$ be reduced to its most simple form.

$$\sqrt{\tfrac{2}{5}\times\tfrac{5}{5}}=\sqrt{\tfrac{10}{25}}=\sqrt{\tfrac{1}{25}\times\tfrac{10}{1}}=\tfrac{1}{5}\sqrt{10}.\ \textit{Ans.}$$

2. Let $\sqrt[3]{\frac{2}{3}}$ be reduced to its most simple form.

$$\sqrt[3]{\tfrac{2}{3}}=\sqrt[3]{\tfrac{2}{3}\times\tfrac{9}{9}}=\sqrt[3]{\tfrac{18}{27}}=\sqrt[3]{\tfrac{1}{27}\times\tfrac{18}{1}}=\tfrac{1}{3}\sqrt[3]{18}.\ \textit{Ans.}$$

3. Let $\sqrt{\frac{2}{7}}$ be reduced to its most simple form.

Ans. $\frac{1}{7}\sqrt{14}.$

4. Let $\sqrt[3]{\frac{2}{5}}$ be reduced to its most simple form.

Ans. $\frac{1}{5}\sqrt[3]{50}.$

5. Let $\sqrt[4]{\frac{1}{2}}$ be reduced to its most simple form.

Ans. $\frac{1}{2}\sqrt[4]{8}.$

EXAMPLES TO EXERCISE THE FOREGOING RULES.

1. What is the most simple form of $\sqrt{125}$? *Ans.* $5\sqrt{5}.$

2. What is the most simple form of $\sqrt{80a^2x^3}$?

Ans $4ax\sqrt{5x}.$

3. What is the most simple form of $\sqrt[3]{189a^4b^3c^2}$?

Ans. $3ab\sqrt[3]{7ac^2}$.

4. What is the most simple form of $7\sqrt{80}$? *Ans.* $28\sqrt{5}$.

5. What is the most simple form of $\frac{3}{7}\sqrt{\frac{4}{5}}$? *Ans.* $\frac{6}{35}\sqrt{5}$.

6. What is the most simple form of $\frac{3}{11}\sqrt{\frac{4}{7}}$? *Ans.* $\frac{6}{77}\sqrt{7}$.

7. Let $\sqrt{96a^2x^3}$ be reduced to its most simple form.

Ans. $4ax\sqrt{6x}$.

8. Let $\frac{3}{7}\sqrt[3]{56x^4+64y^3}$ be reduced to its most simple form.

Ans. $\frac{6}{7}\sqrt[3]{(7x^4+8y^3)}$.

PROBLEM VII.

200. To add surd quantities together.

I. When the radicals are similar, annex the radical part to the sum of the coefficients.

EXAMPLES.

1. Add $7\sqrt{2}$ to $5\sqrt{2}$. *Ans.* $12\sqrt{2}$.

2. Add $5\sqrt{ab}$ to $3\sqrt{ab}$. *Ans.* $8\sqrt{ab}$.

3. Add $a\sqrt{xy}$ to $b\sqrt{xy}$. *Ans.* $(a+b)\sqrt{xy}$.

4. Add $7\sqrt{a^2-y}$ to $y\sqrt{a^2-y}$. *Ans.* $(7+y)\sqrt{a^2-y}$.

II. When the radical parts are dissimilar, make them similar by Art. 197, and proceed as above.

But, if the surd part cannot be made the same in all the quantities, they can only be added by the signs + and —.

5. Add $\sqrt{18}$ and $\sqrt{32}$ together.

First $\sqrt{18}=\sqrt{9\times2}=3\sqrt{2}$.

And $\sqrt{32}=\sqrt{16\times2}=4\sqrt{2}$.

Then $3\sqrt{2}+4\sqrt{2}=7\sqrt{2}$. *Ans.*

6. Required the sum of $\sqrt[3]{375}$ and $\sqrt[3]{192}$.

First $\sqrt[3]{375}=\sqrt[3]{125\times3}=5\sqrt[3]{3}$.

And $\sqrt[3]{192}=\sqrt[3]{64\times3}=4\sqrt[3]{3}$.

Then $5\sqrt[3]{3}+4\sqrt[3]{3}=9\sqrt[3]{3}$. *Ans.*

7. Required the sum of $\sqrt{27}$ and $\sqrt{48}$. *Ans.* $7\sqrt{3}$.

8. Required the sum of $\sqrt{50}$ and $\sqrt{72}$. *Ans.* $11\sqrt{2}$.

9. Find the sum of $\sqrt{180}$ and $\sqrt{405}$. *Ans.* $15\sqrt{5}$.

10. It is required to find the sum of $\sqrt[3]{40}$ and $\sqrt[3]{135}$.
Ans. $5\sqrt[3]{5}$.

11. Find the sum of $4\sqrt[3]{54}$ and $5\sqrt[3]{128}$. *Ans.* $32\sqrt[3]{2}$.

12. Find the sum of $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{1}{32}}$. *Ans.* $\frac{3}{4}\sqrt[3]{2}$.

13. Required the sum of $3\sqrt{a^2b}$ and $5\sqrt{16a^4b}$.
Ans. $(3a+20a^2)\sqrt{b}$.

PROBLEM VIII.

201. To find the difference of surd quantities.

RULE. *When the radicals are, or have been made, similar, annex the common radical part to the difference of the rational parts.*

But, if the quantities have no common surd, they can be subtracted only by changing the sign of the subtrahend.

EXAMPLES.

1. From $\sqrt{320}$ take $\sqrt{80}$.

First $\sqrt{320}=\sqrt{64\times5}=8\sqrt{5}$.
And $\sqrt{80}=\sqrt{16\times5}=4\sqrt{5}$.
Then $8\sqrt{5}-4\sqrt{5}=4\sqrt{5}$. *Ans.*

2. Find the difference between $\sqrt[3]{128}$ and $\sqrt[3]{54}$.

First $\sqrt[3]{128}=\sqrt[3]{64\times2}=4\sqrt[3]{2}$.
And $\sqrt[3]{54}=\sqrt[3]{27\times2}=3\sqrt[3]{2}$.
Then $4\sqrt[3]{2}-3\sqrt[3]{2}=\sqrt[3]{2}$. *Ans.*

3. Required the difference between $2\sqrt{50}$ and $\sqrt{18}$.
Ans. $7\sqrt{2}$.

4. What is the difference between $2\sqrt[3]{320}$ and $3\sqrt[3]{40}$?
Ans. $2\sqrt[3]{5}$.

5. Required the difference of $\sqrt{75}$ and $\sqrt{48}$. *Ans.* $\sqrt{3}$.

6. Required the difference of $\sqrt[3]{256}$ and $\sqrt[3]{32}$.
Ans. $2\sqrt[3]{4}$.

7. Required the difference of $\sqrt[3]{\frac{3}{4}}$ and $\sqrt[3]{\frac{2}{9}}$.
Ans. $\frac{1}{6}\sqrt[3]{6}$.

8. Required the difference of $\sqrt[3]{\frac{3}{5}}$ and $\sqrt[3]{\frac{25}{9}}$.
Ans. $\frac{2}{15}\sqrt[3]{75}$.

9. Find the difference of $\frac{3}{7}\sqrt[3]{a^3b}$ and $\frac{2}{5}\sqrt[3]{a^6b}$.

Ans. $\left(\frac{15a}{35}-\frac{14a^2}{35}\right)\sqrt[3]{b}$.

10. From $\sqrt{4ax^2}$ take $3x\sqrt{9a}$. *Ans.* $-7x\sqrt{a}$.

PROBLEM IX.

202. To multiply surd quantities together.

RULE. *When the surds are of the same kind, find the product of the rational parts, and the product of the surds; and the two joined together, with the common radical sign between them, will give the whole product required, which may be reduced to its most simple form by* Art. 199.

203. If the surds are of different kinds, they must be reduced to a common index, and then multiplied together, as before.

204. Powers and roots of the same quantity are multiplied by adding their exponents.

EXAMPLES.

1. Find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

Here $3\sqrt{8}$

Multiplied by $2\sqrt{6}$

Gives $6\sqrt{48}=6\sqrt{(16\times 3)}=24\sqrt{3}$. *Ans.*

2. Find the product of $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$.

Here $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$

Multiplied by $\underline{\frac{3}{4}\sqrt[3]{\frac{5}{6}}}$

Gives $\frac{3}{8}\sqrt[3]{\frac{5}{9}}=\frac{3}{8}\sqrt[3]{(\frac{5}{9}\times\frac{3}{3})}=\frac{3}{8}\sqrt[3]{(\frac{15}{27})}=$ $\frac{3}{8}\sqrt[3]{(\frac{1}{27}\times\frac{15}{1})}=(\frac{3}{8}\times\frac{1}{3})\sqrt[3]{15}=\frac{1}{8}\sqrt[3]{15}.$ *Ans.*

3. Multiply $2^{\frac{1}{2}}$ by $3^{\frac{1}{3}}$.

Here $2^{\frac{1}{2}}=2^{\frac{3}{6}}=(2^3)^{\frac{1}{6}}=8^{\frac{1}{6}}$.

And $3^{\frac{1}{3}}=3^{\frac{2}{6}}=(3^2)^{\frac{1}{6}}=\underline{9^{\frac{1}{6}}}$

$72^{\frac{1}{6}}$. *Ans.*

4. Multiply $5\sqrt{a}$ by $3\sqrt[3]{a}$.

Here $5\sqrt{a}=5a^{\frac{1}{2}}=5a^{\frac{3}{6}}$.

And $3\sqrt[3]{a}=3a^{\frac{1}{3}}=3a^{\frac{2}{6}}$.

$15a^{\frac{5}{6}}=15\sqrt[6]{a^5}$. *Ans.*

5. Multiply $4\sqrt{12}$ by $3\sqrt{2}$. *Ans.* $24\sqrt{6}$.
6. Multiply $3\sqrt{2}$ by $2\sqrt{8}$. *Ans.* 24.
7. Multiply $\frac{1}{3}\sqrt[3]{4}$ by $\frac{3}{4}\sqrt[3]{12}$. *Ans.* $\frac{1}{2}\sqrt[3]{6}$.
8. Multiply $\frac{5}{3}\sqrt{\frac{3}{8}}$ by $\frac{9}{10}\sqrt{\frac{3}{5}}$. *Ans.* $\frac{9}{4}\sqrt{\frac{1}{10}}$.
9. Multiply $7\sqrt[3]{18}$ by $5\sqrt[3]{4}$. *Ans.* $70\sqrt[3]{9}$.
10. Multiply $\frac{1}{4}\sqrt[3]{6}$ by $\frac{2}{15}\sqrt[3]{17}$. *Ans.* $\frac{1}{30}\sqrt[3]{102}$.
11. Multiply $2a^{\frac{2}{3}}$ by $a^{\frac{4}{3}}$. *Ans.* $2a^2$.
12. Multiply $(a+b)^{\frac{1}{3}}$ by $(a+b)^{\frac{3}{4}}$. *Ans.* $\sqrt[12]{(a+b)^{13}}$.
13. Multiply $x-\sqrt{xy}+y$ by $\sqrt{x}+\sqrt{y}$.

By expressing the surds with fractional indices, we have

$$\begin{array}{llll}
x & -x^{\frac{1}{2}}y^{\frac{1}{2}} & +y. & \\
\underline{x^{\frac{1}{2}}} & \underline{+y^{\frac{1}{2}}} & & \\
x^{\frac{3}{2}} & -xy^{\frac{1}{2}} & +x^{\frac{1}{2}}y & \\
 & \underline{+xy^{\frac{1}{2}}} & \underline{-x^{\frac{1}{2}}y} & \underline{+y^{\frac{3}{2}}} \\
x^{\frac{3}{2}} & & & +y^{\frac{3}{2}}
\end{array}$$

14. Multiply $a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{3}}$.

$$\begin{array}{l}
a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}} \\
a^{\frac{1}{2}}-b^{\frac{1}{3}} \\
\hline
a^3+a^{\frac{5}{2}}b^{\frac{1}{3}}+a^2b^{\frac{2}{3}}+a^{\frac{3}{2}}b+ab^{\frac{4}{3}}+a^{\frac{1}{2}}b^{\frac{5}{3}} \\
\quad -a^{\frac{5}{2}}b^{\frac{1}{3}}-a^2b^{\frac{2}{3}}-a^{\frac{3}{2}}b-ab^{\frac{4}{3}}-a^{\frac{1}{2}}b^{\frac{5}{3}}-b^2 \\
\hline
a^3 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -b^2. \quad \textit{Ans.}
\end{array}$$

15. Multiply $\sqrt{a}+\sqrt{b}+\sqrt{c}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

Ans. $a+b-c+2\sqrt{ab}$.

PROBLEM X.

205. To divide one surd quantity by another.

RULE. *When the surds are of the same kind, find the quotient of the rational parts, and the quotients of the surds, and the two joined together, with the common radical sign between them, will give the whole quotient required.*

But, if the surds are of different kinds, they must be reduced to a common index, and be divided as above.

The quotients of different powers or roots of the same quantity are found by subtracting their indices.

EXAMPLES.

1. Divide $6\sqrt{96}$ by $3\sqrt{8}$.

Here $\dfrac{6\sqrt{96}}{3\sqrt{8}}=2\sqrt{12}=2\sqrt{4\times3}=(2\times2)\sqrt{3}=4\sqrt{3}$. *Ans.*

2. Divide $8\sqrt{108}$ by $2\sqrt{6}$.

Here $\dfrac{8\sqrt{108}}{2\sqrt{6}}=4\sqrt{18}=4\sqrt{9\times2}=(4\times3)\sqrt{2}=12\sqrt{2}$. *Ans.*

3. Divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

Here $\dfrac{8\sqrt[3]{512}}{4\sqrt[3]{2}}=2\sqrt[3]{256}=2\sqrt[3]{64\times4}=8\sqrt[3]{4}$. *Ans.*

4. Divide 12 times the cube root of 280 by 3 times the cube root of 5.

$$\text{Here } \frac{12\sqrt[3]{280}}{3\sqrt[3]{5}}=4\sqrt[3]{56}=4\sqrt[3]{8\times7}=8\sqrt[3]{7}. \quad \textit{Ans.}$$

5. Divide $6\sqrt{54}$ by $3\sqrt{2}$. *Ans.* $6\sqrt{3}$.

6. Divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$. *Ans.* $2\sqrt[3]{4}$.

7. Divide $4\sqrt{50}$ by $2\sqrt{5}$. *Ans.* $2\sqrt{10}$.

8. Divide $6\sqrt[3]{100}$ by $3\sqrt[3]{5}$. *Ans.* $2\sqrt[3]{20}$.

9. Divide $\sqrt{20}+\sqrt{12}$ by $\sqrt{5}+\sqrt{3}$. *Ans.* 2.

10. Divide $32\frac{2}{5}\sqrt{a}$ by $13\frac{3}{4}\sqrt[3]{b}$. *Ans.* $\frac{648}{275}\left(\frac{a^3}{b^2}\right)^{\frac{1}{6}}$.

206. Since the division of surds is performed by subtracting their indices, it is evident that the denominator of any fraction may be taken into the numerator, or the numerator into the denominator, by changing the sign of its index.

EXAMPLES.

1. Let $\frac{1}{a}$ be expressed by a negative index.

$$\frac{1}{a}=\frac{a^{-1}}{1}=a^{-1}.$$

2. Let $\frac{1}{a^n}$ be expressed by a negative index.

$$\frac{1}{a^n}=\frac{a^{-n}}{1}=a^{-n}.$$

3. Let $\frac{b}{a^2}$ be expressed by a negative index.

$$\frac{b}{a^2}=\frac{ba^{-2}}{1}=ba^{-2}.$$

4. Let $\frac{1}{a^2}$ be expressed by a negative index.

$$\frac{1}{a^2}=\frac{a^{-2}}{1}=a^{-2}.$$

5. Let $a^{-\frac{1}{2}}$ be expressed by a positive index.

$$a^{-\frac{1}{2}}=\frac{a^{-\frac{1}{2}}}{1}=\frac{1}{a^{\frac{1}{2}}}.$$

6. Let $\sqrt{\frac{1}{a+x}}$ be expressed by a negative index.

Ans. $(a+x)^{-\frac{1}{2}}$.

7. Let $a(a^2-x^2)^{-\frac{1}{3}}$ be expressed by a positive index.

Ans. $\frac{1}{a(a^2-x^2)^{\frac{1}{3}}}$.

8. What is the value of $\frac{a^m}{a^m}$?

$$\frac{a^m}{a^m}=a^{m-m}=a^0=1.$$

Whence it follows that a^0 is a symbol equivalent to unity; consequently 1 may always be substituted for it. This, however, has been demonstrated in a previous article.

PROBLEM XI.

207. To involve or raise surd quantities to any power.

Let $a^{\frac{h}{s}}$ represent a surd quantity; then, by Art. 204, its square will be

$$a^{\frac{h}{s}}\times a^{\frac{h}{s}}=a^{\frac{h+h}{s}}=a^{\frac{2h}{s}}.$$

Therefore, to involve a surd to any required power, we adopt the following

RULE. *When the surd is a simple quantity, multiply its index by 2 for the square, 3 for the cube, &c., and it will give the power of the surd part, which, being annexed to the proper power of the rational parts, will give the whole power required.*

If the surd be a compound quantity, multiply it by itself the requisite number of times.

EXAMPLES.

1. What is the square of $3a^{\frac{1}{3}}$?

$$3a^{\frac{1}{3}\times\frac{2}{1}}=3a^{\frac{2}{3}}=9\sqrt[3]{a^2}.\quad \textit{Ans.}$$

2. What is the cube of $\frac{2}{3}\sqrt{3}$?

Here $(\frac{2}{3}\sqrt{3})^3=\frac{8}{27}\sqrt{27}=\frac{8}{27}\sqrt{(9\times3)}=\frac{8}{9}\sqrt{3}$. *Ans.*

3. Required the square of $3\sqrt[3]{3}$. *Ans.* $9\sqrt[3]{9}$.

4. Required the cube of $17\sqrt{21}$. *Ans.* $103173\sqrt{21}$.

5. What is the fourth power of $\frac{1}{6}\sqrt{6}$? *Ans.* $\frac{1}{36}$.

6. Required the cube of $\sqrt{3}$. *Ans.* $3\sqrt{3}$.

7. Required the third power of $\frac{1}{2}\sqrt{3}$. *Ans.* $\frac{3}{8}\sqrt{3}$.

8. Required the fourth power of $\frac{1}{2}\sqrt{2}$. *Ans.* $\frac{1}{4}$.

9. What is the *m*th power of $a^{\frac{1}{n}}$? *Ans.* $a^{\frac{m}{n}}$.

10. Required the square of $2+\sqrt{3}$. *Ans.* $7+4\sqrt{3}$.

11. What is the $\frac{r}{s}$th power of $a^{\frac{p}{q}}$? *Ans.* $a^{\frac{pr}{qs}}$.

PROBLEM XII.

208. To find the roots of surd quantities.

RULE. *When the surd is a simple quantity, multiply its index by $\frac{1}{2}$ for the square root, by $\frac{1}{3}$ for the cube root, &c., and it will give the root for the surd part, which being annexed to the root of the rational part, will give the whole root required.*

The truth of this rule may be illustrated by the following

EXAMPLES.

1. What is the cube root of the square root of 64?

The square root of $64=\sqrt{64}=64^{\frac{1}{2}}=8$.

And the cube root of $8=\sqrt[3]{8}=8^{\frac{1}{3}}=2$. *Ans.*

209. The same result would have been obtained if we had multiplied the index ($\frac{1}{2}$) of the given quantity by the index of the required root ($\frac{1}{3}$), the product of which is $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$; and if we had considered this ($\frac{1}{6}$) the index of the root to be extracted of the given quantity 64, the operation would have been thus: $\sqrt[6]{64}=2$. *Ans.*, as before.

2. Required the cube root of the square root of a.

$a^{\frac{1}{2}\times\frac{1}{3}}=a^{\frac{1}{6}}$. *Ans.*

3. Required the fourth root of $\sqrt{3}$. $3^{\frac{1}{2}\times\frac{1}{4}}=3^{\frac{1}{8}}$. *Ans.*

4. What is the square root of $9\sqrt[3]{3}$?

Here $(9\sqrt[3]{3})^{\frac{1}{2}}=9^{\frac{1}{2}}\times3^{\frac{1}{3}\times\frac{1}{2}}=9^{\frac{1}{2}}\times3^{\frac{1}{6}}=3\sqrt[6]{3}$.

5. What is the square root of 10^3?

$10^3=1000$; $\sqrt{1000}=10\sqrt{10}$. *Ans.*

6. What is the cube root of $\frac{27}{64}\sqrt{a}$? *Ans.* $\frac{3}{4}\sqrt[6]{a}$.

7. What is the square root of $\frac{16}{25}a^5$? *Ans.* $\frac{4}{5}a^2\sqrt{a}$.

PROBLEM XIII.

210. To find factors that shall cause any surds to become rational.

I. When the surd is a monomial, multiply it by the same quantity, with an index such as when added to the index of the given quantity will make it a unit.

The quantity $\sqrt{a}$ or $a^{\frac{1}{2}}$ is made rational by multiplying it by $\sqrt{a}$ or $a^{\frac{1}{2}}$.

Thus, $\sqrt{a}\times\sqrt{a}$, or $a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a$.

And it will be rational if $a^{\frac{1}{3}}$ be multiplied by $a^{\frac{2}{3}}$, thus, $a^{\frac{1}{3}}\times a^{\frac{2}{3}}=a$.

Also, if $a^{\frac{4}{5}}$ be multiplied by $a^{\frac{1}{5}}$, it will be rational; thus, $a^{\frac{4}{5}}\times a^{\frac{1}{5}}=a$.

EXAMPLES.

1. What factor will make $x^{\frac{2}{3}}$ rational? *Ans.* $x^{\frac{1}{3}}$.

2. What factor will make $y^{\frac{2}{7}}$ rational? *Ans.* $y^{\frac{5}{7}}$.

3. What factor will cause a^{-3} to become rational?

Ans. a^4.

II. When the surd is a binomial or residual quantity, and both the terms are even roots, to find a factor that will make the quantity rational.

In Art. 158 we have shown that the product of the sum and difference of any two quantities is equal to the difference of their squares; therefore, when one or both of the terms are even roots, we multiply the given binomial or residual by the same quantity, with the sign of one of its terms changed.

NOTE.—It is sometimes necessary to repeat the operation.

EXAMPLES.

1. To find a multiplier or factor that shall make $4+\sqrt{5}$ rational.

$$\begin{array}{ll}
\text{Given surd,} & 4+\sqrt{5} \\
\text{Multiplier,} & 4-\sqrt{5} \\
\hline
 & 16+4\sqrt{5} \\
 & \quad\; -4\sqrt{5}-5 \\
\hline
\text{Product,} & 16 \qquad\quad -5=11 \text{ rational quantity.}
\end{array}$$

2. Find a factor that shall make $\sqrt{a}+\sqrt{b}$ rational.

$$\begin{array}{l}
\sqrt{a}+\sqrt{b} \\
\sqrt{a}-\sqrt{b} \\
\hline
a+\sqrt{ab} \\
\quad -\sqrt{ab}-b \\
\hline
a \qquad\quad -b \text{ rational quantity.}
\end{array}$$

3. What factor will make $1+\sqrt{3}$ rational?

$$\begin{array}{l}
1+\sqrt{3} \\
1-\sqrt{3} \\
\hline
1+\sqrt{3} \\
\; -\sqrt{3}-3 \\
\hline
1 \qquad\quad -3=-2 \text{ rational quantity.}
\end{array}$$

4. What factor will make $\sqrt{5}-\sqrt{1}$ rational?

$$\begin{array}{ll}
\sqrt{5}-\sqrt{1} & \\
\sqrt{5}+\sqrt{1} & \\ \hline
5-\sqrt{5} & \\
\quad +\sqrt{5} & -1 \\ \hline
5 & -1=4 \text{ rational quantity.}
\end{array}$$

5. Find multipliers that shall make $\sqrt[4]{5}+\sqrt[4]{3}$ rational.

$$\begin{array}{ll}
\sqrt[4]{5}+\sqrt[4]{3} & \\
\sqrt[4]{5}-\sqrt[4]{3} & \\ \hline
\sqrt{5}+\sqrt[4]{15} & \\
\quad -\sqrt[4]{15} & -\sqrt{3} \\ \hline
\sqrt{5} & -\sqrt{3} \\
\sqrt{5} & +\sqrt{3} \\ \hline
5-\sqrt{15} & \\
\quad +\sqrt{15} & -3 \\ \hline
5 & -3=2 \text{ rational quantity.}
\end{array}$$

6. What multiplier will make $\sqrt{5}-\sqrt{x}$ rational?

$$\begin{array}{ll}
\sqrt{5}-\sqrt{x} & \\
\sqrt{5}+\sqrt{x} & \\ \hline
5-\sqrt{5x} & \\
\quad +\sqrt{5x} & -x \\ \hline
5 & -x \text{ rational quantity.}
\end{array}$$

III. A trinomial surd may be rendered rational by changing the sign of one of its terms for the multiplier.

EXAMPLES.

1. To find multipliers that shall make $\sqrt{7}+\sqrt{3}-\sqrt{2}$ rational.

$$\begin{array}{l}
\sqrt{7}+\sqrt{3}-\sqrt{2} \\
\sqrt{7}+\sqrt{3}+\sqrt{2} \\
\hline
7+\sqrt{21}-\sqrt{14} \\
\quad +\sqrt{21}+3-\sqrt{6} \\
\qquad\qquad +\sqrt{14}+\sqrt{6}-2 \\
\hline
8+2\sqrt{21} \\
-8+2\sqrt{21} \\
\hline
-64-16\sqrt{21} \\
\quad +16\sqrt{21}+84 \\
\hline
84-64=20 \text{ rational quantity.}
\end{array}$$

2. Find a factor that will make $\sqrt{8}-\sqrt{1}-\sqrt{3}$ rational.

$$\begin{array}{l}
\sqrt{8}-\sqrt{1}-\sqrt{3} \\
\sqrt{8}+\sqrt{1}+\sqrt{3} \\
\hline
8-\sqrt{8}-\sqrt{24} \\
\quad +\sqrt{8}-1-\sqrt{3} \\
\qquad\qquad +\sqrt{24}-\sqrt{3}-3 \\
\hline
4-2\sqrt{3} \\
4+2\sqrt{3} \\
\hline
16-8\sqrt{3} \\
\quad +8\sqrt{3}-12 \\
\hline
16-12=4 \text{ rational quantity.}
\end{array}$$

QUESTIONS FOR EXERCISE.

1. Find a multiplier that shall make $\sqrt{5}-\sqrt{2}$ rational.
Ans. $\sqrt{5}+\sqrt{2}$.

2. Find a multiplier that shall make $\sqrt{7}+\sqrt{6}$ rational.
Ans. $\sqrt{7}-\sqrt{6}$.

3. Find a multiplier that shall make $\sqrt{10}-\sqrt{2}$ rational.
Ans. $\sqrt{10}+\sqrt{2}$.

4. Find multipliers that shall make $\sqrt{a}+\sqrt{b}+\sqrt{c}$ rational.

Ans. $\sqrt{a}-\sqrt{b}-\sqrt{c}$, and $(a-b-c+2\sqrt{bc})$.

5. Find multipliers that shall make $\sqrt[4]{3}-\sqrt[4]{1}$ rational.

Ans. $(\sqrt[4]{3}+\sqrt[4]{1})(\sqrt{3}+\sqrt{1})$.

PROBLEM XIV.

ART. 211. To reduce a fraction, whose denominator is a surd, to another that shall have a rational denominator, without changing its value.

RULE 1. *When the proposed fraction is a simple one, multiply each of its terms by the denominator.*

2. *If it be a compound surd, find such a multiplier by the last* Art. *as will make the denominator rational, then multiply both the numerator and denominator by it.*

EXAMPLES.

1. Reduce $\frac{b}{\sqrt{a}}$ to a fraction whose denominator shall be rational.

$$\frac{b}{\sqrt{a}}\times\frac{\sqrt{a}}{\sqrt{a}}=\frac{b\sqrt{a}}{a}. \quad \textit{Ans.}$$

2 Reduce $\frac{b}{\sqrt[3]{a}}$ to a fraction whose denominator shall be rational.

$$\frac{b}{\sqrt[3]{a}}\times\frac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}}=\frac{b\sqrt[3]{a^2}}{a}. \quad \textit{Ans.}$$

3. Reduce the fraction $\frac{2}{\sqrt{5}}$ to another whose denominator shall be rational.

$$\frac{2}{\sqrt{5}}=\frac{2}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}=\frac{2\sqrt{5}}{5}. \quad \textit{Ans.}$$

4. Reduce $\frac{3}{\sqrt{5}-\sqrt{2}}$ to a fraction whose denominator shall be rational.

Here $\frac{3}{\sqrt{5}-\sqrt{2}}=\frac{3}{\sqrt{5}-\sqrt{2}}\times\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{3\sqrt{5}+3\sqrt{2}}{5-2}=$

$$\frac{3\sqrt{5}+3\sqrt{2}}{3}=\frac{\sqrt{5}+\sqrt{2}}{1}=\sqrt{5}+\sqrt{2}. \quad \textit{Ans.}$$

5. Extract the square root of $\frac{5}{8}$.

Here $\sqrt{\frac{5}{8}}=\frac{\sqrt{5}}{\sqrt{8}}=\frac{\sqrt{5}}{\sqrt{8}}\times\frac{\sqrt{8}}{\sqrt{8}}=\frac{\sqrt{40}}{\sqrt{64}}=\frac{2\sqrt{10}}{8}=\frac{\sqrt{10}}{4}$. *Ans.*

6. Reduce $\frac{\sqrt{2}}{3-\sqrt{2}}$ to a fraction whose denominator shall be rational.

Here $\frac{\sqrt{2}}{3-\sqrt{2}}=\frac{\sqrt{2}}{3-\sqrt{2}}\times\frac{3+\sqrt{2}}{3+\sqrt{2}}=\frac{3\sqrt{2}+2}{9-2}=\frac{3\sqrt{2}+2}{7}=$

$\frac{2}{7}+\frac{3}{7}\sqrt{2}$. *Ans.*

7. Reduce $\frac{\sqrt{7}}{\sqrt{5}+\sqrt{3}}$ to a fraction that shall have a rational denominator. *Ans.* $\frac{\sqrt{35}-\sqrt{21}}{2}$.

8. Reduce $\frac{\sqrt{3}}{3-\sqrt{1}}$ to an equivalent fraction having a rational denominator. *Ans.* $\frac{\sqrt{3}}{2}$.

9. Reduce the fraction $\frac{5}{\sqrt{5}-\sqrt{2}}$ to an equivalent fraction having a rational denominator. *Ans.* $\frac{5\sqrt{5}+5\sqrt{2}}{3}$.

10. Reduce the fraction $\frac{10}{\sqrt{5}+\sqrt{3}}$ to an equivalent fraction having a rational denominator. *Ans.* $\frac{5\sqrt{5}-5\sqrt{3}}{1}$.

11. Reduce $\frac{1}{\sqrt{5}+\sqrt{7}}$ to a fraction that shall have a rational denominator. *Ans.* $\frac{\sqrt{5}-\sqrt{7}}{-2}$.

12. Reduce the fraction $\frac{3}{\sqrt{5}-\sqrt{2}}$ to an equivalent fraction that shall have a rational denominator.

Ans. $\frac{3\sqrt{5}+3\sqrt{2}}{3}=\sqrt{5}+\sqrt{2}.$

PROBLEM XV.

212. To change a binomial, or residual surd, into a general surd.

RULE. *Involve the given binomial, or residual, to a power corresponding with that denoted by the surd; then write the radical sign of the same root over it.*

EXAMPLES.

1. It is required to reduce $2+\sqrt{3}$ to a general surd.

Here, $(2+\sqrt{3})^2=4+4\sqrt{3}+3=7+4\sqrt{3}.$

Therefore, $2+\sqrt{3}=\sqrt{(7+4\sqrt{3})}.$

2. Reduce $\sqrt{2}+\sqrt{3}$ to a general surd.

Here, $(\sqrt{2}+\sqrt{3})^2=2+2\sqrt{6}+3=5+2\sqrt{6}.$

Therefore, $\sqrt{2}+\sqrt{3}=\sqrt{(5+2\sqrt{6})}.$

3. Reduce $\sqrt[3]{2}+\sqrt[3]{4}$ to a general surd.

Here, $(\sqrt[3]{2}+\sqrt[3]{4})^3=6+6\sqrt[3]{2}+6\sqrt[3]{4}.$

Therefore, $\sqrt[3]{2}+\sqrt[3]{4}=\sqrt[3]{6(1+\sqrt[3]{2}+\sqrt[3]{4})}.$

4. Let $3-\sqrt{5}$ be reduced to a general surd.

Ans. $\sqrt{(14-6\sqrt{5})}.$

5. Let $\sqrt{2}+2\sqrt{6}$ be changed to a general surd.

Ans. $\sqrt{(26+8\sqrt{3})}.$

6. It is required to change $4-\sqrt{7}$ to a general surd.

Ans. $\sqrt{(23-8\sqrt{7})}.$

7. Let $7\sqrt[3]{3}-3\sqrt[3]{9}$ be changed to a general surd.

Ans. $\sqrt[3]{(786-1323\sqrt[3]{3}+567\sqrt[3]{9})}.$

PROBLEM XVI.

TO EXTRACT THE SQUARE ROOT OF A BINOMIAL SURD.

213. A binomial surd is one in which one of the terms, at least, is irrational; as $a\pm\sqrt{b}$, or $\sqrt{a}\pm\sqrt{b}$.

To extract the square root of $a+\sqrt{b}$, we put

$$\sqrt{(a+\sqrt{b})}=m+n.$$

And $$\sqrt{(a-\sqrt{b})}=m-n.$$

By squaring both of these equations,

We have $$a+\sqrt{b}=m^2+2mn+n^2.$$

And $$a-\sqrt{b}=m^2-2mn+n^2.$$

By addition, $$2a\quad=2m^2\quad+2n^2.$$

And $$a=m^2+n^2.$$

Multiplying the two first equations together,

We have $$\sqrt{(a+\sqrt{b})}\times\sqrt{(a-\sqrt{b})}=(m+n)\times(m-n).$$

And $$\sqrt{(a^2-b)}=m^2-n^2.$$

Having both the sum and difference of m^2 and n^2, we obtain, by addition and subtraction, the following equations:

$$m^2=\frac{a+\sqrt{(a^2-b)}}{2}, \text{ and } n^2=\frac{a-\sqrt{(a^2-b)}}{2}.$$

Therefore, $$m=\sqrt{\left(\frac{a+\sqrt{(a^2-b)}}{2}\right)},$$

$$\text{and } n=\sqrt{\left(\frac{a-\sqrt{(a^2-b)}}{2}\right)}.$$

Consequently, $$\sqrt{(a+\sqrt{b})}=\sqrt{\left(\frac{a+\sqrt{(a^2-b)}}{2}\right)}+\sqrt{\left(\frac{a-\sqrt{(a^2-b)}}{2}\right)}.$$

And $$\sqrt{(a-\sqrt{b})}=\sqrt{\left(\frac{a+\sqrt{(a^2-b)}}{2}\right)}-\sqrt{\left(\frac{a-\sqrt{(a^2-b)}}{2}\right)}.$$

It is certain that both a and $\sqrt{(a^2-b)}$ must be rational, in order that the expressions within the parentheses may be rational, in which case each of the above values will be either two surds, or a rational and a surd.

The above formulæ will apply to any particular values for a and b; observing that if b be negative, the signs of b in the formulæ must be changed.

EXAMPLES.

1. What is the square root of $11+\sqrt{72}$?

Here, $a=11$, and $b=72$. Therefore,

$$\sqrt{\left(\frac{a+\sqrt{(a^2-b)}}{2}\right)}=\sqrt{\left(\frac{11+\sqrt{(11^2-72)}}{2}\right)}=3.$$

And $\sqrt{\left(\frac{a-\sqrt{(a^2-b)}}{2}\right)}=\sqrt{\left(\frac{11-\sqrt{(11^2-72)}}{2}\right)}=\sqrt{2}.$

Therefore, $\sqrt{(11+\sqrt{72})}=3+\sqrt{2}.$

2. What is the square root of $10-\sqrt{96}$?

Let $a=10$, and $b=96$.

Then $\sqrt{\left(\frac{a+\sqrt{(a^2-b)}}{2}\right)}=\sqrt{\left(\frac{10+\sqrt{(10^2-96)}}{2}\right)}=\sqrt{6}.$

And $\sqrt{\left(\frac{a-\sqrt{(a^2-b)}}{2}\right)}=\sqrt{\left(\frac{10-\sqrt{(10^2-96)}}{2}\right)}=2.$

Therefore, $\sqrt{(10-\sqrt{96})}=\sqrt{6}-2.$

3. What is the square root of $6+\sqrt{20}$? *Ans.* $1+\sqrt{5}$.

4. What is the square root of $6+2\sqrt{5}$? *Ans.* $\sqrt{5}+1$.

5. What is the square root of $12+2\sqrt{35}$? *Ans.* $\sqrt{5}+\sqrt{7}$.

6. Required the square root of $36\pm10\sqrt{11}$.

Ans. $5\pm\sqrt{11}$.

7. What is the square root of $7-2\sqrt{10}$? *Ans.* $\sqrt{5}-\sqrt{2}$.

8. What is the square root of $1+4\sqrt{-3}$?

Ans. $2+\sqrt{-3}$, or $2-\sqrt{-3}$.

SECTION XVI.

IMAGINARY QUANTITIES.

Art. 214. As every algebraical symbol hitherto considered, whether it be affected with the sign $+$ or $-$, when raised to an even power gives a positive result, it follows that no even root of a *negative* quantity can be either positive or negative. The even roots of negative quantities having, therefore, no symbolical representation in accordance with the views of Algebra, so far as we have yet considered it, can only be *indicated* or *expressed* by means of the radical sign, or corresponding fractional index. Hence arises a new species of symbolical expressions, called *Imaginary* or *Impossible* Quantities.

Thus the square root of $-a^2$ is neither $+a$ nor $-a$, but is written $\sqrt{-a^2}$, and is equivalent to $\sqrt{a^2\times(-1)}=\sqrt{a^2}\sqrt{-1}$ $=\pm a\sqrt{-1}$, which is said to be impossible, or imaginary, in consequence of involving the symbol $\sqrt{-1}$.

By Art. 78 we learn that the product of real quantities, that have like signs, is always *plus;* and, if the signs are unlike, the product is *minus*. We, therefore, infer, that the product of two imaginary quantities, that have the same sign, is equal to *minus* the square root of their product, considering them as real quantities.

Hence,

$$(+\sqrt{-a})(+\sqrt{-a})=-\sqrt{a^2}=-a.$$
$$(-\sqrt{-a})(-\sqrt{-a})=-\sqrt{a^2}=-a.$$
$$(+\sqrt{-a})(+\sqrt{-b})=-\sqrt{ab}.$$
$$(-\sqrt{-a})(-\sqrt{-b})=-\sqrt{ab}.$$

215. If the two imaginary quantities have different signs, then, it is evident, their product will be equal to *plus* the square root of their product, considering them as real.

Thus,

$$(+\sqrt{-a})(-\sqrt{-b})=+\sqrt{ab}.$$

EXAMPLES.

1. Multiply $4\sqrt{-3}$ by $2\sqrt{-2}$.

$$4\sqrt{-3}\times2\sqrt{-2}=-8\sqrt{6}.$$

2. Multiply $4+\sqrt{-3}$ by $3-\sqrt{-5}$.

$$\begin{array}{r}
4+\sqrt{-3}\phantom{-4\sqrt{-5}+\sqrt{15}} \\
3-\sqrt{-5}\phantom{-4\sqrt{-5}+\sqrt{15}} \\
\hline
12+3\sqrt{-3}\phantom{-4\sqrt{-5}+\sqrt{15}} \\
-4\sqrt{-5}+\sqrt{15} \\
\hline
12+3\sqrt{-3}-4\sqrt{-5}+\sqrt{15}.
\end{array}$$

3. Multiply $3\sqrt{-1}$ by $7\sqrt{-8}$. *Ans.* $-21\sqrt{8}$.

4. Multiply $-7\sqrt{-4}$ by $-3\sqrt{-3}$. *Ans.* $-21\sqrt{12}$.

5. Multiply $4+\sqrt{-7}$ by $\sqrt{-2}$. *Ans.* $4\sqrt{-2}-\sqrt{14}$.

216. If one imaginary be divided by another, having the same signs, the quotient is equal to plus the square root.

But, if the imaginaries have different signs, it is evident that their quotient will be equal to minus the square root of their quotient.

EXAMPLES.

6. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$. *Ans.* $3\sqrt{\frac{3}{4}}$.

7. Divide $2\sqrt{-10}$ by $-5\sqrt{-2}$. *Ans.* $-\frac{2}{5}\sqrt{5}$.

8. Divide $-\sqrt{-1}$ by $-7\sqrt{-3}$. *Ans.* $+\frac{1}{7\sqrt{3}}$.

9. Divide $+\sqrt{-a}$ by $+\sqrt{-b}$. *Ans.* $+\sqrt{\frac{a}{b}}$.

10. Divide $-\sqrt{-a}$ by $-\sqrt{-b}$. *Ans.* $+\sqrt{\frac{a}{b}}$.

11. Divide $4+\sqrt{-2}$ by $2-\sqrt{-2}$. *Ans.* $1+\sqrt{-2}$.

12. Divide $1+\sqrt{-1}$ by $1-\sqrt{-1}$. *Ans.* $\sqrt{-1}$.

13. Divide $2\sqrt{-7}$ by $-3\sqrt{-5}$. *Ans.* $-\frac{2}{3}\sqrt{\frac{7}{5}}$.

SECTION XVII.

QUADRATIC EQUATIONS, OR EQUATIONS OF THE SECOND DEGREE.

ART. **217.** A quadratic equation is one in which the unknown quantity rises to the second power.

Quadratics are of two kinds: those which contain only the square of the unknown quantity are called pure quadratics, and those which contain both the first and second powers of the unknown quantity are called affected quadratic equations.

The following are examples of pure quadratics:

EXAMPLES.

1. Given $4x^2-7=29$ to find x.

Conditions, $4x^2-7=29$.
Transposing, $4x^2=29+7=36$.
Dividing, $x^2=9$.
Extracting square root, $x=\pm 3$.

2. Given $ax^2+b=c$ to find x.

Conditions, $ax^2+b=c$.
Transposing, $ax^2=c-b$.
Dividing, $x^2=\frac{c-b}{a}$.
Extracting square root, $x=\pm\sqrt{\frac{c-b}{a}}$.

Hence, to find the value of the unknown term, we have the following

RULE. *Transpose and reduce the equation, so that the unknown quantity may be positive, and the first member of the equation. Divide both members of the equation by the coefficient of the unknown quantity; then extract the square root of both members.*

3. Given $5x^2+5=3x^2+55$ to find x.

Conditions, $5x^2+5=3x^2+55$.

Transposing, $5x^2-3x^2=55-5$.

Reducing, $2x^2=50$.

Dividing, $x^2=25$.

Extracting square root, $x=\pm 5$.

4. Given $2x^2+8=3x^2-28$ to find x.

Conditions, $3x^2-28=2x^2+8$.

Transposing, $3x^2-2x^2=28+8$.

Reducing, $x^2=36$.

Extracting square root, $x=\pm 6$.

5. Given $7x^2-5=3x^2+11$ to find x. *Ans.* $x=\pm 2$.

6. Given $4x^2+15=7x^2-417$ to find x. *Ans.* $x=\pm 12$.

7. Given $3x^2+7=\frac{5x^2}{4}+35$ to find x. *Ans.* $x=\pm 4$.

8. Given $ax^2+n=m-c$ to find x. *Ans.* $x=\pm\sqrt{\frac{m-c-n}{a}}$.

9. Given $x^2-ab=d$ to find x. *Ans.* $x=\pm\sqrt{d+ab}$.

10. A lady bought a silk dress for £8 15*s.*, and the number of shillings she paid per yard was, to the number of yards, as 4 to 7. How many yards did she purchase for her dress, and what was the price per yard?

Let x = the number of shillings paid per yard.

Then $\frac{7x}{4}$ = the number of yards.

And the price of the whole, $\frac{7x^2}{4}=175$ shillings.

Clearing of fractions, $7x^2=700$.

Dividing, $x^2=100$.

Extracting the square root, $x=10$*s.*, price per yd.

Therefore, $\frac{7x}{4}=17\frac{1}{2}$ yards. *Ans.*

11. I have 10 acres of land. If it were a square field, what would be the length of one of its sides? *Ans.* 40 rods.

12. A and B lay out money on speculation; the amount of A's stock and gain is \$27, and he gains as much per cent. on his stock as B lays out. B's gain is \$32; and it appears that A gains twice as much per cent. as B. Required the capital of each. *Ans.* A's capital, \$15; B's, \$80.

13. There are two square fields, the larger of which contains 25,600 square rods more than the other, and the ratio of their sides is as 5 to 3. Required the contents of each.

Ans. Contents of the larger, 40,000 square rods.
Contents of the smaller, 14,400 square rods.

14. I have three square house-lots, of equal size; if I were to add 193 square rods to their contents, they would be equivalent to a square lot whose sides would measure each 25 rods. Required the length of each of the sides of my three house-lots.

Ans. 12 rods each.

15. A farmer has a square field, and the number of rods round it is $\frac{1}{10}$ the number of square rods of its contents. Required the number of acres in the field. *Ans.* 10 acres.

16. John Smith has a field, which is a right-angled parallelogram; its sides are in the ratio of 4 to 3; a diagonal, passing from one corner to its opposite, is 100 rods. Required the contents of the field. *Ans.* 30 acres.

17. Two workmen, A and B, engage to work for a certain number of days, at different rates. At the end of the time, A, who had been absent 4 days, received 75 shillings; but B, who had been absent 7 days, received only 48 shillings. Now, if B had been absent only 4 days, and A 7 days, they would have received exactly alike. How many days were they engaged for, how many did each work, and what had each per day?

Ans. They were engaged to work 19 days. A worked 15, and B 12 days; A received 5 shillings, and B 4 shillings per day.

18. Two numbers are to each other as 4 to 5, and the sum of their cubes is 1512. What are those numbers?

Ans. 8 and 10.

19. A bushel measure contains $2150\frac{2}{5}$ cubic inches, and I wish to make a box that shall contain 50 bushels. Its length is to be to its breadth as 3 to 1, and its height $\frac{3}{4}$ its breadth. What are its dimensions?

Ans. Length 108.84+, breadth 36.28+, and height 27.21+ inches.

20. What must be the dimensions of a cubical box that shall contain 100 bushels?

Ans. Height, length, and breadth, 59.9+ inches.

21. Two numbers are to each other as 3 to 7, and the difference of their cubes is 2528. What are those numbers?

Ans. 6 and 14.

22. Bought a house-lot for $5184. Its length is to its breadth as 3 to 1. I gave as many dollars per square rod as the lot is rods in breadth. What were the dimensions of the lot?

Ans. 36 rods long, 12 rods wide.

PROBLEMS.

23. Let m be divided into two parts, whose squares shall be to each other as n to p.

Let $x =$ the greater.

And $m-x =$ the less.

Then $x^2 : (m-x)^2 :: n : p$.

Multiplying extremes, $px^2=n(m-x)^2$.

Evolution, $x\sqrt{p}=\pm\sqrt{n}(m-x)$.

Reducing, $x\sqrt{p}=m\sqrt{n}-x\sqrt{n}$.

Transposing, $x\sqrt{p}+x\sqrt{n}=m\sqrt{n}$.

Dividing, $x=\dfrac{m\sqrt{n}}{\sqrt{p}+\sqrt{n}}$ the greater.

Subtracting, $m-\dfrac{m\sqrt{n}}{\sqrt{p}+\sqrt{n}}=\dfrac{m\sqrt{p}}{\sqrt{p}+\sqrt{n}}$ the less.

If we take the minus sign, we have

$$x\sqrt{p}=-\sqrt{n}(m-x).$$

Multiplying, $x\sqrt{p}=-m\sqrt{n}+x\sqrt{n}.$

Transposing, $x\sqrt{p}-x\sqrt{n}=-m\sqrt{n}$

Changing signs, $x\sqrt{n}-x\sqrt{p}=m\sqrt{n}.$

Dividing, $x=\dfrac{m\sqrt{n}.}{\sqrt{n}-\sqrt{p}}$ the greater,

Subtracting, $m-\dfrac{m\sqrt{n}}{\sqrt{n}-\sqrt{p}}=\dfrac{-m\sqrt{p}}{\sqrt{n}-\sqrt{p}}$ the less.

24. Divide 18 into two such parts that the square of the larger part shall be 25 times the square of the less.

Let x = the larger; then $18-x$ = the less.

Then we have $x^2 : (18-x)^2 :: 25 : 1.$

Multiplying extremes, $x^2=25(18-x)^2.$

Evolution, $x=5(18-x).$

Multiplying, $x=90-5x.$

Transposition, $6x=90.$

Dividing, $x=15$, the larger.

$18-15=3$, the less.

VERIFICATION.

$15^2=25(3)^2.$

Involving, $225=225.$

THE THEORY OF THE LIGHTS AND ATTRACTION.

218. To apply the foregoing problems, we premise the following principles of Natural Philosophy.

1. The intensity of light emanating from any luminous body is inversely as the square of the distance from that body; that is, if the earth were twice the distance from the sun that it now is, it would receive only one-fourth part of the light and heat that it now does; and, if it were removed to ten times the distance, it would have only one-hundredth part of the light and heat.

2. The quantity of light emanating from a celestial body is directly as the square of its diameter.

Hence, if the earth were four times the diameter of the moon, an inhabitant of that luminary would receive sixteen times as much light from the earth as he would receive from the moon if he were on the earth.

3. The laws of attraction are similar to those of light, for all bodies attract each other inversely as the squares of the distances from their centre, and directly as the masses of matter which compose those bodies.

APPLICATION OF THE ABOVE PRINCIPLES.

25. The moon is 240,000 miles from the earth, and the quantity of matter in the earth is 80 times that of the moon. At what distance from the earth, in a direct line towards the moon, must a body be placed to be equally attracted by each, so that it will remain at rest as it respects those bodies?

Let d = the distance between the moon and earth.
e = the quantity of matter in the earth.
m = the quantity in the moon.
x = the distance from the earth to the point required.

Then $d-x$ = the distance from the moon.

We have then the following proposition:

As $x^2 : (d-x)^2 :: e : m.$
Therefore, $mx^2=e(d-x)^2.$
By evolution, $x\sqrt{m}=\sqrt{e}(d-x).$
Reducing, $x\sqrt{m}=d\sqrt{e}-x\sqrt{e}.$
Transposing, $x\sqrt{m}+x\sqrt{e}=d\sqrt{e}.$
Dividing, $x=\dfrac{d\sqrt{e}}{\sqrt{m}+\sqrt{e}}.$

Substituting the value of d, e and m, we have

$x=\dfrac{240{,}000\sqrt{80}}{\sqrt{80}+\sqrt{1}}=\dfrac{2146624.8}{8.94427+1}=215865.4$ miles, = the distance from the earth.

$240,000-215865.4=24134.6$ miles, $=$ the distance from the moon.

If we take the negative sign, we shall find the point *beyond* the moon where the attraction of the two bodies will be equal.

Taking the minus sign, $x\sqrt{m}=-\sqrt{e}(d-x)$.

Reducing, $x\sqrt{m}=-d\sqrt{e}+x\sqrt{e}$.

Transposing, $x\sqrt{e}-x\sqrt{m}=d\sqrt{e}$.

Dividing, $x=\dfrac{d\sqrt{e}}{\sqrt{e}-\sqrt{m}}$.

Substituting the values of d, e and m, we have

$x=\dfrac{2146624.8}{\sqrt{80}-\sqrt{1}}=270,210$ miles from the earth's centre, and, therefore, $270,210-240,000=30,210$ miles beyond the moon.

26. Required the distance from the earth, in a direction towards the sun, where a body would remain at rest, the distance of the earth being 95,000,000 miles from the sun, and the quantity of matter in the sun being 333,928 times greater than that of the earth.

Let S represent the quantity of matter in the sun, E the quantity of matter in the earth, and D the distance between the earth and sun, and x the required distance from the sun.

Then, substituting these letters for those in question 23, we have the following formula:

$$x=\frac{d\sqrt{s}}{\sqrt{s}+\sqrt{1}}=$$

$$\frac{95,000,000\sqrt{333,928}}{\sqrt{333,928}+\sqrt{1}}=94,835,885;$$

$95,000,000-94,835,885=164,115$ miles. *Ans*

27. The diameter of Venus is 7700 miles, its distance from the sun is 68,000,000; the diameter of the earth is 7912 miles, and its distance from the sun, as stated above, is 95,000,000 miles. How much greater, therefore, is the intensity of light at

Venus than at the earth, and what is the comparative quantity that each receives from the sun?

Ans. The intensity of light at Venus is 1.95+ times greater than at the earth. Venus receives from the sun 1.84+ times more light than the earth.

28. Mercury is 37,000,000 miles from the sun. How much greater, therefore, is the intensity of light and heat at Mercury than at the earth? *Ans.* $6\frac{811}{1369}$ times.

29. Jupiter is 490,000,000 miles from the sun, and its diameter is 89,000 miles. Saturn is 900,000,000 miles from the sun, and its diameter is 79,000 miles. How much more light, therefore, do we receive from Jupiter than from Saturn, when they are in opposition to the sun?

Let $a =$ the distance of Jupiter from the sun.
$b =$ the diameter of Jupiter.
$c =$ his distance from the earth.
$d =$ the distance of Saturn from the sun.
$e =$ the diameter of Saturn.
$h =$ his distance from the earth.

The distance of these planets from the earth is obtained by subtracting the earth's distance from the sun from their distance from the sun.

The surface of Jupiter is to the surface of Saturn as the squares of their diameters; and as the quantity of light which a planet receives from the sun is as the square of its diameter directly, and inversely as the squares of its distance from the sun,

Therefore, if $b^2 =$ the surface of Jupiter,
and $e^2 =$ the surface of Saturn,
and a and d their respective distances from the sun,

then the intensity of light at Saturn will be to the intensity of light at Jupiter as $\frac{e^2}{d^2}$ is to $\frac{b^2}{a^2}$. And as the light which each of these planets gives to the earth is in intensity inversely as the squares of their distances from the earth,

therefore, if $\frac{e^2}{d^2} =$ the quantity of light at Saturn, and $\frac{b^2}{a^2} =$

the quantity of light at Jupiter, then $\frac{e^2}{d^2h^2}$ = the quantity of light which Saturn gives to the earth, and $\frac{b^2}{a^2c^2}$ = the quantity which Jupiter gives.

Therefore, to find how much more light we receive from Jupiter than from Saturn, we use the following proportion:

$$\frac{e^2}{d^2h^2} : \frac{b^2}{a^2c^2} :: 1 : x.$$

Therefore, $$x=\frac{d^2h^2b^2}{e^2a^2c^2}.$$

If we substitute for these letters their numerical values, we shall have

$$x=\frac{900^2\times805^2\times89^2}{79^2\times490^2\times395^2}=$$

$$\frac{810000\times648025\times7921}{6241\times240100\times156025}=17.7+. \quad \textit{Ans.}$$

That is, we receive more than seventeen times as much light from Jupiter as we do from Saturn.

In the above operation, we have cancelled the ciphers in the distances and diameters of the planets.

AFFECTED QUADRATIC EQUATIONS.

219. An affected quadratic equation is one containing the first power of the unknown quantity in one term, and the square of that quantity in another term.

Every equation of this kind, having any real or positive root, will fall, when properly reduced, under one of the four following forms:

1. $x^2+ax=\ \ b$, where $x=-\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}+b\right)}$.
2. $x^2-ax=\ \ b$, where $x=+\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}+b\right)}$.
3. $x^2+ax=-b$, where $x=-\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}-b\right)}$.
4. $x^2-ax=-b$, where $x=+\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}-b\right)}$.

220. No exact root can be taken of a binomial; but, if the first term of a binomial be a square of the unknown quantity, and the second term the quantity itself, with 1, or any other quantity, for its coefficient, the square of half the coefficient of the second term, added to the binomial, will make the whole quantity an exact square. This may be illustrated by the following examples.

Let x^2+4x be the binomial, then 2 is half the coefficient of the second term, and its square is $2\times2=4$. This we add to the binomial, and the result is x^2+4x+4, and this quantity is an exact square, and its root, by Art. 183, is $x+2$.

If the binomial be x^2+ax, and we add to it the square of half the coefficient of x, $\frac{a^2}{4}$, the sum will be $x^2+ax+\frac{a^2}{4}$, the exact root of which is $x+\frac{a}{2}$.

Again, if the binomial be x^2-3abx, we have only to add the square of half the coefficient of x, which is $\frac{9a^2b^2}{4}$, to the binomial, and the sum will be an exact square, $x^2-3abx+\frac{9a^2b^2}{4}$.

For
$$\left(x^2-3abx+\frac{9a^2b^2}{4}\right)^{\frac{1}{2}}=x-\frac{3ab}{2}.$$

221. If, therefore, there be any binomial whose first term is an even power of the unknown quantity, and the second term half that power, and we add the square of half the coefficient of the second term to the binomial, the result will be an exact square.

222. To solve an affected quadratic equation, we adopt the following

RULE. *Bring all the unknown terms to one side of the equation, and the known terms to the other, observing so to arrange them that the term which contains the square of the unknown quantity shall be positive, and stand first in the equation, and the term which contains the first power of the unknown quantity the second term of the equation.*

Divide each side of the equation by the coefficient of the unknown square.

Add the square of half the coefficient of the second term to each side of the equation, and the unknown side will be a complete square.

Extract the square root of each side of the equation, and from the result the value of the unknown quantity may be obtained.

Given $x^2+8x=84$ to find the values of x.

Here, by the question, $x^2+8x=84$.

Completing the squares, $x^2+8x+16=84+16=100$.

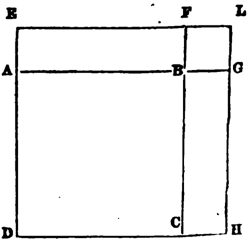
Extracting the square root, $x+4=10$.

Whence, $x=10-4$.

And, $x=6$. *Ans.*

In solving this question, we first add the square of half of 8, that is, 16, to both sides of the equation; we then extract the square root of $x^2+8x+16$, and find the result to be $x+4$, and the square root of $100=10$. Therefore, $x+4=10$, that is, $x=10-4=6$. *Ans.*

223. It may also be demonstrated, by the following diagram, that if the square of half the coefficient of the second term be added to the first member of an equation, it will be a complete square.



Let x represent one side of the square ABCD; then x^2 will represent this square. To this square we must add $8x$, and this quantity must be applied equally to the two sides AB and BC, or the figure would not be a square. Therefore $4x$, which is half of $8x$, will be applied to either side. If this quantity, $4x$, be divided by x, the quotient, 4, will represent either of the distances EA or BG. Having added the two equal parallelograms EABF and BGHC to the square ABCD, we find our figure needs the small square FBGL to complete the square. The contents of this must be equal to the product of

FB and BG, that is, 4 multiplied by 4, or the square of 4 = 16; but 4 is half the coefficient of the second term. We add this quantity to x^2+8x, and the sum is $x^2+8x+16$, and its square root is $x+4$, by Art. 182.

224. A quadratic may be solved by the following

RULE. *Having transposed the unknown terms to one side of the equation, and the known to the other, multiply each side by 4 times the coefficient of the square of the unknown quantity.*

Add the square of the coefficient of the first power of the unknown quantity to both sides of the equation, and the unknown side will then be a complete square.

Extract the root of both members, and the value of the unknown quantity is obtained as before.

EXAMPLES.

1. Given $3x^2+4x-7=88$ to find the values of x.

Conditions, $3x^2+4x-7=88$.

Transposing, $3x^2+4x=88+7=95$.

Multiplying by 4 times 3, $36x^2+48x=1140$.

Completing the square, $36x^2+48x+16=1140+16=1156$.

Evolving, $6x+4=\pm 34$.

Transposing, $6x=\pm 34-4=30$, or -38.

Dividing, $x=5$, or $-6\frac{1}{3}$.

2. Given $2x^2-10x+7=-5$ to find the values of x.

Conditions, $2x^2-10x+7=-5$.

Transposing, $2x^2-10x=-5-7=-12$.

Multiplying by 4 times 2, $16x^2-80x=-96$.

Completing the square, $16x^2-80x+100=-96+100=4$.

Evolving, $4x-10=\pm 2$.

Transposing, $4x=\pm 2+10=12$, or 8.

Dividing, $x=3$, or 2.

3. Given $3x^2+5x-8=34$ to find the values of x.

Ans. $x=3$, or $-4\frac{2}{3}$.

4. Given $x^2+6x+4=22-x$ to find the values of x.

Ans. $x=2$, or -9.

5. Given $8x^2-7x+6=171$ to find the values of x.

Ans. $x=5$, or $-\frac{33}{8}$.

6. Given $\frac{175x-350}{x}+10x-20=175$ to find the values of x.

Ans. $x=7$, or -5.

7. Given $x^2-6x+12=4$ to find the values of x.

Ans. $x=4$, or 2.

8. Given $8x^2+32x=360$ to find the values of x.

Conditions, $8x^2+32x=360.$

Dividing, $x^2+4x=45.$

Completing the square, $x^2+4x+4=45+4=49.$

Evolving, $x+2=\pm 7.$

Transposing, $x=\pm 7-2=5$, or -9.

9. Given $x^2-8x+50=98$ to find the values of x.

Conditions, $x^2-8x+50=98.$

Transposing, $x^2-8x=98-50=48.$

Completing the square, $x^2-8x+16=48+16=64.$

Evolving, $x-4=\pm 8.$

Transposing, $x=\pm 8+4=12$, or -4.

10. Given $x^2+ax=b$ to find the values of x.

Conditions, $x^2+ax=b.$

Completing the square, $x^2+ax+\frac{a^2}{4}=b+\frac{a^2}{4}.$

Evolving, $x+\frac{a}{2}=\pm\sqrt{\left(b+\frac{a^2}{4}\right)}.$

Transposing, $x=\pm\sqrt{\left(b+\frac{a^2}{4}\right)}-\frac{a}{2}.$

11. Given $3x^2-3x+6=5\frac{1}{3}$ to find the values of x.

Conditions, $3x^2-3x+6=5\frac{1}{3}.$

Transposing, $3x^2-3x=5\frac{1}{3}-6=-\frac{2}{3}$.
Reducing, $x^2-x=-\frac{2}{9}$.
Completing the square, $x^2-x+\frac{1}{4}=-\frac{2}{9}+\frac{1}{4}=+\frac{1}{36}$.
Evolving, $x-\frac{1}{2}=\pm\frac{1}{6}$.
Transposing, $x=\pm\frac{1}{6}+\frac{1}{2}=\frac{2}{3}$, or $\frac{1}{3}$.

12. Given $\frac{x^2}{2}-\frac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$ to find the values of x.

Conditions, $\frac{x^2}{2}-\frac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$.

Transposing, $\frac{x^2}{2}-\frac{x}{3}=42\frac{2}{3}-20\frac{1}{2}=22\frac{1}{6}$.

Clearing of fractions, $x^2-\frac{2x}{3}=44\frac{1}{3}$.

Completing the square, $x^2-\frac{2x}{3}+\frac{1}{9}=44\frac{1}{3}+\frac{1}{9}=\frac{400}{9}$.

Evolving, $x-\frac{1}{3}=\pm\frac{20}{3}=\pm6\frac{2}{3}$.

Transposing, $x=\pm6\frac{2}{3}+\frac{1}{3}=7$, or $-6\frac{1}{3}$.

13. Given $ax^2+bx=c$ to find the value of x.
Conditions, $ax^2+bx=c$.

Dividing, $x^2+\frac{bx}{a}=\frac{c}{a}$.

Completing the square, $x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}$.

Evolving and transposing, $x=\pm\sqrt{\left(\frac{c}{a}+\frac{b^2}{4a^2}\right)}-\frac{b}{2a}$.

14. Given $ax^2-bx+c=d$ to find the values of x.
Conditions, $ax^2-bx+c=d$.
Transposing, $ax^2-bx=d-c$.

Dividing, $x^2-\frac{bx}{a}=\frac{d-c}{a}$.

Completing the square, $x^2-\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{d-c}{a}+\frac{b^2}{4a^2}$.

Evolving, $x-\frac{b}{2a}=\pm\sqrt{\left(\frac{d-c}{a}+\frac{b^2}{4a^2}\right)}$.

Transposing, $x=\frac{b}{2a}\pm\sqrt{\left(\frac{d-c}{a}+\frac{b^2}{4a^2}\right)}$.

Reducing, $x=\frac{b}{2a}\pm\frac{1}{2a}\sqrt{[4a(d-c)+b^2]}$.

225. If the equation contains two powers of the unknown quantity, and the exponent of the one is double that of the other, it may be resolved like a quadratic. Thus,

15. Given $x^4+4x^2=117$ to find the values of x.

Conditions, $x^4+4x^2=117$.
Completing the square, $x^4+4x^2+4=117+4=121$.
Evolving, $x^2+2=\pm 11$.
Transposing, $x^2=\pm 11-2=9$, or -13.
Evolving, $x=3$, or $\sqrt{-13}$.

16. Given $x^6-6x^3=16$ to find the values of x.

Conditions, $x^6-6x^3=16$.
Completing the square, $x^6-6x^3+9=16+9=25$.
Evolving, $x^3-3=\pm 5$.
Transposing, $x^3=\pm 5+3=8$, or -2.
Evolving, $x=2$, or $\sqrt[3]{-2}$.

17. Given $\frac{x}{2}-\frac{x^{\frac{1}{2}}}{3}=22\frac{1}{6}$ to find the values of x.

Conditions, $\frac{x}{2}-\frac{x^{\frac{1}{2}}}{3}=22\frac{1}{6}$.

Clearing of fractions, $x-\frac{2x^{\frac{1}{2}}}{3}=44\frac{1}{3}$.

Completing the square, $x-\frac{2x^{\frac{1}{2}}}{3}+\frac{1}{9}=44\frac{1}{3}+\frac{1}{9}=\frac{400}{9}$.

Evolving, $x^{\frac{1}{2}}-\frac{1}{3}=\pm\frac{20}{3}$.

Transposing, $x^{\frac{1}{2}}=\pm\frac{20}{3}+\frac{1}{3}=\frac{21}{3}=7$, or $-\frac{19}{3}$.

Involving, $x=49$, or $+\frac{361}{9}$.

18. Given $3x^{2n}-2x^{n}=25$ to find the value of x.

Conditions, $3x^{2n}-2x^{n}=25$.

Dividing, $x^{2n}-\frac{2x^{n}}{3}=\frac{25}{3}$.

Completing the square, $x^{2n}-\frac{2x^{n}}{3}+\frac{1}{9}=\frac{25}{3}+\frac{1}{9}=\frac{76}{9}$.

Evolving, $x^{n}-\frac{1}{3}=\frac{\sqrt{76}}{3}=\frac{2\sqrt{19}}{3}$.

Transposing, $x^{n}=\frac{1}{3}+\frac{2\sqrt{19}}{3}=\frac{1+2\sqrt{19}}{3}$.

Evolving, $x=\left(\frac{1+2\sqrt{19}}{3}\right)^{\frac{1}{n}}$.

19. Given $\sqrt{4x+16}=12$ to find the value of x.

Conditions, $\sqrt{4x+16}=12$.
Squaring both sides of the equation, $4x+16=144$.
Transposing, $4x=144-16=128$.
Dividing, $x=32$.

20. Given $\sqrt[3]{2x+3}+4=7$ to find the value of x.

Conditions, $\sqrt[3]{2x+3}+4=7$.

Transposing, $\sqrt[3]{2x+3}=7-4=3$.

Involving both sides, $2x+3=27$.

Transposing, $2x=27-3=24.$

Dividing, $x=12.$

21. **Given** $\sqrt{12+x}=2+\sqrt{x}$ **to find the value of** x.

Conditions, $\sqrt{12+x}=2+\sqrt{x}.$

Squaring both sides, $12+x=4+4\sqrt{x}+x.$

Transposing, &c., $8=4\sqrt{x}.$

Dividing, $2=\sqrt{x}.$

Involving, $4=x.$

22. **Given** $\sqrt{x+40}=10-\sqrt{x}$ **to find the value of** x.

Conditions, $\sqrt{x+40}=10-\sqrt{x}.$

Squaring both sides, $x+40=100-20\sqrt{x}+x.$

Transposing and reducing, $20\sqrt{x}=60.$

Dividing, $\sqrt{x}=3.$

Involving, $x=9.$

23. **Given** $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$ **to find the value of** x.

Conditions, $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}.$

Involving, $x-a=x-\sqrt{ax}+\frac{a}{4}.$

Transposing, $\sqrt{ax}=a+\frac{a}{4}=\frac{5a}{4}.$

Involving, $ax=\frac{25a^2}{16}.$

Dividing by a**,** $x=\frac{25a}{16}.$

24. **Given** $8x^{\frac{4}{3}}-\frac{5x^{\frac{8}{3}}}{2}=-592$ **to find the values of** x.

Conditions, $8x^{\frac{4}{3}}-\frac{5x^{\frac{8}{3}}}{2}=-592.$

Changing the signs, &c., $\frac{5x^{\frac{8}{3}}}{2}-3x^{\frac{4}{3}}=592.$

Multiplying by $\frac{2}{5}$**,** $x^{\frac{8}{3}}-\frac{6x^{\frac{4}{3}}}{5}=\frac{1184}{5}.$

Completing the square, $x^{\frac{8}{3}}-\frac{6x^{\frac{4}{3}}}{5}+\frac{9}{25}=\frac{1184}{5}+\frac{9}{25}=\frac{5929}{25}$.

Extracting the root, $x^{\frac{4}{3}}-\frac{3}{5}=\pm\frac{77}{5}$.

Transposing, $x^{\frac{4}{3}}=\pm\frac{77}{5}+\frac{3}{5}=16$, or $-\frac{74}{5}$.

Evolving, $x=8$, or $\left(-\frac{74}{5}\right)^{\frac{3}{4}}$.

25. Given $\sqrt{2x+1}+2\sqrt{x}=\frac{21}{\sqrt{2x+1}}$ to find the values of x.

Conditions, $\sqrt{2x+1}+2\sqrt{x}=\frac{21}{\sqrt{2x+1}}$.

Clearing of fractions, $2x+1+2\sqrt{2x^2+x}=21$.

Transposition, $2\sqrt{2x^2+x}=20-2x$.

Division, $\sqrt{2x^2+x}=10-x$.

Squaring both sides, $2x^2+x=100-20x+x^2$.

Transposing, $x^2+21x=100$.

Completing the squares, $x^2+21x+\frac{441}{4}=100+\frac{441}{4}=\frac{841}{4}$.

Evolution, $x+\frac{21}{2}=\pm\frac{29}{2}$.

Transposition, $x=\pm\frac{29}{2}-\frac{21}{2}=4$, or -25.

26. Given $2\sqrt{x-a}+3\sqrt{2x}=\frac{7a+5x}{\sqrt{x-a}}$ to find the values of x.

Conditions, $2\sqrt{x-a}+3\sqrt{2x}=\frac{7a+5x}{\sqrt{x-a}}$.

Multiplying, $2x-2a+3\sqrt{2x^2-2ax}=7a+5x$.

Transposing, $3\sqrt{2x^2-2ax}=9a+3x$

Dividing, $\sqrt{2x^2-2ax}=3a+x$.

Involving, $2x^2-2ax=9a^2+6ax+x^2$.

Transposing, $x^2-8ax=9a^2$.

Completing the squares, $x^2-8ax+16a^2=25a^2$.

Evolving, $x-4a=\pm5a$.

Transposing, $x=\pm5a+4a=9a$, or $-a$.

27. Given $x+5=\sqrt{x+5}+6$ to find the values of x.

Ans. $x=4$, or -1.

28. Given $\sqrt{5x+10}=\sqrt{5x}+2$ to find the value of x.

Conditions, $\sqrt{5x+10}=\sqrt{5x}+2$.

Squaring both sides, $5x+10=5x+4\sqrt{5x}+4$.

Transposing, &c., $6=4\sqrt{5x}$.

Dividing, $3=2\sqrt{5x}$.

Involving, $9=20x$.

Dividing, &c., $x=\frac{9}{20}$.

29. Given $\frac{\sqrt{x}+2a}{\sqrt{x}+b}=\frac{\sqrt{x}+4a}{\sqrt{x}+3b}$ to find the value of x.

Conditions, $\frac{\sqrt{x}+2a}{\sqrt{x}+b}=\frac{\sqrt{x}+4a}{\sqrt{x}+3b}$.

Multiplying both sides of the equation by $\sqrt{x}+b$ and $\sqrt{x}+3b$, we have

$$x+(2a+3b)\times\sqrt{x}+6ab=x+(4a+b)\times\sqrt{x}+4ab.$$

Reducing, &c., $(2a-2b)\times\sqrt{x}=2ab$.

Dividing, $\sqrt{x}=\frac{ab}{a-b}$.

Involving, $x=\left(\frac{ab}{a-b}\right)^2$.

30. Given $\frac{1}{x}+\frac{1}{a}=\sqrt{\frac{1}{a^2}+\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}}}$ to find the value of x.

Conditions, $\frac{1}{x}+\frac{1}{a}=\sqrt{\frac{1}{a^2}+\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}}}$.

Squaring both sides, $\frac{1}{x^2}+\frac{2}{ax}+\frac{1}{a^2}=\frac{1}{a^2}+\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}}$.

Transposing, &c., $\frac{1}{x^2}+\frac{2}{ax}=\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}}.$

Multiplying by x, $\frac{1}{x}+\frac{2}{a}=\sqrt{\frac{4}{a^2}+\frac{9}{x^2}}.$

Squaring both sides, $\frac{1}{x^2}+\frac{4}{ax}+\frac{4}{a^2}=\frac{4}{a^2}+\frac{9}{x^2}.$

Reducing, &c., $\frac{4}{ax}=\frac{8}{x^2}.$

Dividing, &c., $\frac{1}{a}=\frac{2}{x}.$

Transposing, &c., $x=2a.$

31. Given $x=\sqrt{a^2+x\sqrt{b^2+x^2}}-a$ to find the value of x.

Ans. $x=\frac{b^2-4a^2}{4a}.$

32. Given $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$ to find the value of x.

Ans. $x=\frac{1}{1-a}.$

33. Given $x^2+12x-16=92$ to find the values of x.

Ans. $x=6$, or -18.

34. Given $x^2-3x=10$ to find the values of x.

Ans. $x=5$, or -2.

35. Given $x^2-x+3=45$ to find the values of x.

Ans. $x=7$, or -6.

36. Given $5x^2+x=4$ to find the values of x.

Ans. $x=\frac{4}{5}$, or -1.

37. Given $2x^2-x=21$ to find the values of x.

Ans. $x=\frac{7}{2}$, or -3.

38. Given $5x^2+6x-3=60$ to find the values of x.

Ans. $x=3$, or $-\frac{21}{5}$.

39. Given $(x-12)(x+2)=0$ to find the values of x.

Ans. $x=12$, or -2.

40. Given $3x^2-14x+15=0$ to find the values of x.

Ans. $x=3$, or $1\frac{2}{3}$.

41. Given $ax^2-bx=c$ to find the values of x.

Ans. $x=\frac{b\pm\sqrt{(b^2+4ac)}}{2a}$.

42. Given $4x^2-6x=108$ to find the values of x.

Ans. $x=6$, or $-4\frac{1}{2}$.

43. Given $4x-\frac{14-x}{x+1}=14$ to find the values of x.

Ans. $x=4$, or $-\frac{7}{4}$.

44. Given $\frac{10}{x}-\frac{14-2x}{x^2}=\frac{22}{9}$ to find the values of x.

Ans $x=3$, or $\frac{21}{11}$.

45. Given $x+\sqrt{5x+10}=8$ to find the values of x

Ans. $x=18$, or 3.

46. Given $x+\sqrt{10x+6}=9$ to find the values of x.

Ans. $x=25$, or 3.

47. Given $3x^2+2x-9=76$ to find the value of x.

Ans. $x=5$, or $-5\frac{2}{3}$.

48. Given $x^2-10x=-25$ to find the value of x.

Ans. $x=5$.

49. Given $3x^2-x-140=0$ to find the value of x.

Ans. $x=7$, or $\frac{20}{3}$.

50. Given $5x^2+\frac{7x}{2}=7x^2-51$ to find the value of x.

Ans. $x=6$, or $-5\frac{1}{2}$.

51. Given $2x^2-\frac{4x-4}{3}=7x$ to find the value of x.

Ans. $x=4$, or $\frac{1}{6}$.

52. Given $\frac{x^2}{5}+20x=3x^2-80$ to find the value of x.

Ans. $x=10$, or $-2\frac{6}{7}$.

53. If $x^2+8x=65$, what are the two values of x?

Ans. $x=5$, or -13.

54. If $6x^2-x=92$, what are the two values of x?

Ans. $x=4$, or $-\frac{23}{6}$.

55. If $3x^2+4x=340$, what are the two values of x?

Ans. $x=10$, or $-11\frac{1}{3}$.

56. If $x^2-10x=-21$, what are the two values of x?

Ans. $x=7$, or 3.

57. If $5x^2-\frac{x}{2}=78$, what are the two values of x?

Ans. $x=4$, or $-3\frac{9}{10}$.

58. If $11x^2-100x=-201$, what are the two values of x?

Ans. $x=3$, or $6\frac{1}{11}$.

59. If $3x^2-17x=2x^2+84$, what are the two values of x?

Ans. $x=21$, or -4.

60. Given $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$ to find the values of x. *Ans.* $x=9$, or -12.

61. Given $9x+\sqrt{16x^2+36x^3}=15x^2-4$ to find the values of x. *Ans.* $x=\frac{4}{3}$, or $-\frac{1}{3}$.

62. Given $x=\frac{12+8x^{\frac{1}{2}}}{x-5}$ to find the values of x.

Ans. $x=9$, or 4.

63. Given $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=\frac{x^2}{a}$ to find the value of x.

Ans. $x=\pm a\sqrt{\frac{1\pm\sqrt{5}}{2}}$.

64. Given $x-1=2+\frac{2}{x^{\frac{1}{2}}}$ to find the values of x.

Ans. $x=4$, or 1.

65. Given $\sqrt[3]{x^3-a^3}=x-b$ to find the values of x.

Ans. $x=\frac{b}{2}\pm\sqrt{\frac{4a^3-b^3}{12b}}$.

66. Given $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$ to find the values of x.

Ans. $x=4$, or $\frac{64}{9}$.

(*See Key,* p. 119.)

67. Given $\sqrt{x^3}-2\sqrt{x}-x=0\sqrt{x}$ to find the values of x.

Ans. $x=4$, or 1.

68. Given $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$ to find the values of x.

Ans. $x=2$, or -3.

69. Given $\frac{x}{2}=22\frac{1}{6}+\frac{\sqrt{x}}{3}$ to find the values of x.

Ans. $x=49$, or $\frac{361}{9}$.

70. Given $\dfrac{\frac{3\sqrt{x}}{5}-2}{x-5}-\frac{1}{20}=0$ to find the values of x.

Ans. $x=49$, or 25.

71. Given $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$ to find the values of x.

Ans. $x=243$, or $-28^{\frac{5}{3}}$.

72. Given $x^3-x^{\frac{3}{2}}=56$ to find the values of x.

Ans. $x=4$, or $\sqrt[3]{49}$

73. Given $\sqrt{5+x}+\sqrt{x}=\dfrac{15}{\sqrt{5+x}}$ to find the value of x.

Ans. $x=4$.

74. Given $\sqrt{x+12}+\sqrt[4]{x+12}=6$ to find the values of x.

Ans. 4, or 69.

75. Given $x^n-2ax^{\frac{n}{2}}=b$ to find the values of x.

Ans. $x=(a\pm\sqrt{a^2+b})^{\frac{2}{n}}$.

76. Given $3x^{\frac{4}{3}}-\dfrac{5x^{\frac{8}{3}}}{2}=-592$ to find the values of x.

Ans. $x=8$, or $\left(-\frac{74}{5}\right)^{\frac{3}{4}}$.

PROBLEMS.

1. A merchant bought a number of pieces of two kinds of silk, for £92 3*s*. There were as many pieces bought of each kind, and as many shillings paid per yard for them, as a piece of that kind contained yards. Now, two pieces, one of each kind, together measured 19 yards. How many yards were there in each?

Let $x =$ the number of yards in one piece; it will also equal the number of pieces, and also the number of shillings per yard; and $19-x =$ the number of yards in the other piece.

Therefore, $x^3+(19-x)^3=$ the value of both kinds.

And $x^3+(19-x)^3=1843.$

Or $57x^2-1083x+6859=1843.$

By transposition, $57x^2-1083x=-5016.$

Or $x^2-19x=-88.$

Completing the square, $x^2-19x+\frac{361}{4}=\frac{361}{4}-88=\frac{9}{4}.$

Evolution, $x-\frac{19}{2}=\pm\frac{3}{2}.$

$$x=\pm\frac{3}{2}+\frac{19}{2}=11 \text{ or } 8.$$

$$19-x=8 \text{ or } 11.$$

Both values answer the conditions of the question; therefore there were 11 yards in one, and 8 in the other.

2. The plate of a looking-glass is 18 inches by 12, and is to be framed with a frame all parts of which are of equal width, and whose area is to be equal to that of the glass. Required the width of the frame. *Ans.* 3 inches.

3. A grazier bought as many sheep as cost him £60, out of which he reserved 15, and sold the remainder for £54, gaining two shillings a head on them. How many sheep did he buy, and what was the price of each?

Ans. 75 sheep, at 16 shillings each.

4. A merchant sold a quantity of flour for $39, and gained as much per cent, as the flour cost him. What was the price of the flour? *Ans.* $30.

5. There are two numbers, whose difference is 9, and whose sum multiplied by the greater is 266. What are those numbers? *Ans.* 14 and 5.

6. A and B gained, by trade $18; A's money was in the

firm 12 months, and he received, for his principal and gain, \$26. B's money, which was \$30, was in the firm 16 months. What money did A put into the firm? *Ans.* \$20.

7. A merchant bought a quantity of flour for \$72, and he found that if he had bought 6 barrels more for the same money, he would have paid \$1 less for each barrel. How many barrels did he buy, and what was the price of each?

Ans. He bought 18 barrels, at \$4 per barrel.

8. A square court-yard has a gravel-walk around it. The side of the court wants 2 yards of being 6 times the breadth of the gravel-walk, and the number of square yards in the walk exceeds the number of yards in the perimeter of the court by 164 yards. Required the area of the court.

Ans. 256 square yards.

9. Given $\frac{1+x^3}{(1+x)^3}=\frac{1}{3}$ to find the values of x.

Ans. $x=2$, or $\frac{1}{2}$.

10. Given $x^4-2x^3+x=132$ to find the values of x.

Ans. $x=\frac{1\pm\sqrt{-43}}{2}$.

11. Given $9x+\sqrt{16x^2+36x^3}=15x^2-4$ to find the values of x. *Ans.* $x=\frac{4}{3}$, or $-\frac{1}{3}$.

12. It is required to find two numbers, the first of which may be to the second as the second is to 16, and the sum of the squares of the numbers may be equal to 225.

Ans. 9 and 12

QUADRATICS WITH TWO OR MORE UNKNOWN TERMS.

1. Given $\left.\begin{array}{l} x+y=10 \\ \text{And } xy=16 \end{array}\right\}$ to find the values of x and y.

(1.) First equation, $x+y=10$.

(2.) Second equation, $xy=16$.

(3.) Squaring the 1st, $x^2+2xy+y^2=100$.

(4.) Multiplying (2) by 4, $4xy=64$.

(5.) Subtracting 4th from 3d, $x^2-2xy+y^2=36$.
(6.) Evolving 5th, $x-y=\pm 6$.
(7.) The 1st, $x+y=10$.
(8.) Adding 6th and 7th, $2x=16$, or 4.
(9.) Subtracting 6th from 7th, $2y=4$, or 16.
(10.) Dividing the 8th by 2, $x=8$, or 2.
(11.) Dividing the 9th by 2, $y=2$, or 8.

Hence, $x=8$ or 2, and $y=2$ or 8.

This method may be adopted whenever the sum and product of two unknown quantities are given.

2. Given $\left.\begin{array}{l} x-y=3 \\ \text{And } xy=10 \end{array}\right\}$ to find the values of x and y.

(1.) First condition, $x-y=3$.
(2.) Second condition, $xy=10$.
(3.) Squaring 1st, $x^2-2xy+y^2=9$.
(4.) Multiplying 2d by 4, $4xy=40$.
(5.) Adding 3d and 4th, $x^2+2xy+y^2=49$.
(6.) Evolving the 5th, $x+y=\pm 7$.
(7.) The 1st, $x-y=3$.
(8.) Adding 6th and 7th, $2x=10$, or -4.
(9.) Dividing 8th by 2, $x=5$, or -2.
(10.) Subtracting 7th from 6th, $2y=4$, or -10.
(11.) Dividing 10th by 2, $y=2$, or -5.

Hence, $x=5$ or -2, and $y=2$ or -5.

We may proceed in the same manner whenever the difference and product of two unknown quantities are given.

3. Given $\left.\begin{array}{l} x+y=20 \\ \text{And } x^2+y^2=208 \end{array}\right\}$ to find the values of x and y.

(1.) First equation, $x+y=20$.
(2.) Second equation, $x^2+y^2=208$.
(3.) 2d multiplied by 2, $2x^2+2y^2=416$.

(4.) Square of the 1st, $x^2+2xy+y^2=400$.
(5.) Subtracting 4th from 3d, $x^2-2xy+y^2=16$.
(6.) Evolving 5th, $x-y=\pm 4$.
(7.) First equation, $x+y=20$.
(8.) Sum of 6th and 7th, $2x=24$, or 16.
(9.) Half of the 8th, $x=12$, or 8.
(10.) Subtracting 6th from 7th, $2y=16$, or 24.
(11.) Half of 10th, $y=\ 8$, or 12.

Hence, $x=12$, or 8; $y=8$, or 12.

4. Given $\left.\begin{array}{l} x-y=3 \\ \text{And } x^2+y^2=117 \end{array}\right\}$ to find the values of x and y.

(1.) First equation, $x-y=3$.
(2.) Second equation, $x^2+y^2=117$.
(3.) The 2d multiplied by 2, $2x^2+2y^2=234$.
(4.) Square of the 1st, $x^2-2xy+y^2=9$.
(5.) Subtracting 4th from 3d, $x^2+2xy+y^2=225$.
(6.) Evolving the 5th, $x+y=\pm 15$.
(7.) The 1st, $x-y=3$.
(8.) Sum of the 6th and 7th, $2x=18$, or -12.
(9.) Dividing 8th by 2, $x=9$, or -6.
(10.) Subtracting 7th from 6th, $2y=12$, or -18.
(11.) Dividing 10th by 2, $y=6$, or -9.

Hence, $x=9$, or -6; $y=6$, or -9.

5. Given $\left.\begin{array}{l} \sqrt{x^2+y^2}=10 \\ \text{And } x^2-y^2=28 \end{array}\right\}$ to find the values of x and y.

(1.) First equation, $\sqrt{x^2+y^2}=10$.
(2.) Second equation, $x^2-y^2=28$.
(3.) Square of the 1st, $x^2+y^2=100$.
(4.) Sum of 2d and 3d, $2x^2=128$.
(5.) Half the 4th, $x^2=64$.
(6.) Square root of 5th, $x=8$.

(7.) Subtract 2d from 3d, $2y^2=72$.

(8.) Half the 7th, $y^2=36$.

(9.) Square root of 8th, $y=6$.

Hence, $x=8$, and $y=6$.

6. Given $\left.\begin{array}{l} x+y=5 \\ \text{And } x^3+y^3=35 \end{array}\right\}$ to find the values of x and y.

(1.) First equation, $x+y=5$.

(2.) Second equation, $x^3+y^3=35$.

(3.) Square of the 1st, $x^2+2xy+y^2=25$.

(4.) The 2d divided by the 1st $x^2-xy+y^2=7$.

(5.) Subtracting 4th from 3d, $3xy=18$.

(6.) Dividing 5th by 3, $xy=6$.

(7.) The 4th, $x^2-xy+y^2=7$.

(8.) The 6th, $xy=6$.

(9.) Subtracting 6th from 7th, $x^2-2xy+y^2=1$.

(10.) Evolving the 9th, $x-y=1$.

(11.) The 1st, $x+y=5$.

(12.) Sum of 10th and 11th, $2x=6$.

(13.) Half of 12th, $x=3$.

(14.) Subtracting 10th from 11th, $2y=4$.

(15.) Half of 14th, $y=2$.

Hence, $x=3$, and $y=2$.

7. Given $\left.\begin{array}{l} x^2+y^2=20 \\ \text{And } x^2-y^2=12 \end{array}\right\}$ to find the values of x and y.

Ans. $x=4$; $y=2$.

8. Given $\left.\begin{array}{l} x+y=6 \\ \text{And } x^2+y^2=26 \end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, and $y=1$.

9. Given $\left.\begin{array}{l} x^2+y^2=74 \\ \text{And } x-y=2 \end{array}\right\}$ to find the values of x and y.

Ans. $x=7$, and $y=5$.

10. Given $\left.\begin{array}{l} x^2+y^2=149 \\ \text{And } x+y=17 \end{array}\right\}$ to find the values of x and y.

Ans. $x=10$, and $y=7$.

11. Given $\left.\begin{array}{l} x^2-y^2=85 \\ x+y=17 \end{array}\right\}$ to find the values of x and y.
And

Ans. $x=11$, and $y=6$.

12. Given $\left.\begin{array}{l} x-y=2 \\ x^3-y^3=98 \end{array}\right\}$ to find the values of x and y.
And

Ans. $x=5$, and $y=3$.

13. Given $\left.\begin{array}{l} 10x+y=3xy \\ y-x=2 \end{array}\right\}$ to find the values of x and y.
And

Ans. $x=2$ or $-\frac{1}{3}$, and $y=4$ or $+\frac{5}{3}$.

EXAMPLES OF ONE OR MORE UNKNOWN TERMS.

1. A says to B, The sum of our money is 18 dollars; B replies, But if twice the number of your dollars were multiplied by mine, the product would be $154. How many dollars had each? *Ans.* A had $7, and B had $11.

2. The difference of two numbers is 5, and the sum of their squares is 193. What are those numbers? *Ans.* 12 and 7.

3. A and B have each a small field, each of which is an exact square, and it requires 200 rods of fence to enclose both. The contents of these fields are 1300 square rods. What is the value of each, at $2.25 per square rod?

Ans. A's field, $900; B's, $2025.

4. A lady wishes to purchase a carpet for each of her square parlors, one of which is 3 feet longer than the other, and it will require 85 square yards for both rooms. Mr. Ames has good carpeting, which is 40 inches wide, which he will sell at $1.75 per yard. What will it cost the lady to carpet each of her rooms? *Ans.* For the larger room, $77.17½; smaller, $56.70.

5. There are two piles of wood, each of which is a perfect cube; the sum of their lengths is 20 feet, and their contents are 2240 cubic feet. What is the value of each pile, at $6.25 per cord?

Ans. Value of the larger pile, $84.37½; the smaller, $25.

6. There are two square buildings, that are paved with stones a foot square each. The perimeter of the larger building ex-

ceeds that of the smaller by 48 feet, and both their pavements together contain 2120 stones. What are the lengths respectively? *Ans.* 26 and 38 feet.

7. A sets out from Boston for Portland, the distance being 105 miles. B sets out at the same time from Portland for Boston. A arrives in Portland in 9 hours, B arrives in Boston in 16 hours, after they meet. In what time does each perform the journey? *Ans.* A in 21 hours; B in 28 hours.

8. Divide 60 into two such parts that their product shall be to the difference of their squares as 2 to 3. *Ans.* 40 and 20.

9. There are two numbers whose product is 77, and the difference of whose squares is to the square of their difference as 9 to 2. Required the numbers. *Ans.* 11 and 7.

10. I have two house-lots, the contents of which are 225 square rods, and the area of the less is to the area of the larger as 9 to 16. Required the contents of each lot.
Ans. 81 square rods in the less, and 144 in the larger.

11. The product of two numbers is 48, and the difference of their cubes is to the cube of their difference as 37 to 1. Required the numbers. *Ans.* 8 and 6.

12. There are two numbers whose product is 196, and if the greater be divided by the less the quotient is 4. What are those numbers? *Ans.* 28 and 7.

13. A, B and C, can perform a piece of work in a certain time; A can perform it in 6 hours, B in 15 hours, and C in 10 hours. How long would it take them all to perform it?
Ans. 3 hours.

14. A grazier bought a certain number of oxen for $240, and having lost 3, he sold the remainder at $8 a head more than they cost him, thus gaining $59 by his bargain. What number did he buy? *Ans.* 16.

15. The paving of two court-yards cost £205; a square yard of each cost $\frac{1}{4}$ as many shillings as there were yards in a side

of the other; and a side of the greater and less together measure 41 yards. Required the length of a side of each.

Ans. 25 and 16 yards.

16. Divide 145 into two such parts, that the sum of their square roots shall be 17. *Ans.* 81 and 64.

17. Sold an ox for $56, and gained as much per cent. as the ox cost. What was paid for him? *Ans.* $40.

18. Divide the number 14 into two parts, so that the sum of their cubes shall be 728. *Ans.* 8 and 6.

19. My farm is a rectangle, and the length is twice its breadth; but, having enlarged it two rods on all sides, I find its contents increased 496 square rods. Of how many acres does my farm at present consist? *Ans.* 23 acres, 16 rods.

20. There are two numbers whose product added to the sum of their squares is 109, but the difference of whose squares is 24. Required those numbers. *Ans.* 5 and 7.

21. What number is that to which if 40 be added, and the square root extracted, this root shall be less than the original quantity by 16? *Ans.* 24.

22. Two gentlemen, A and B, speaking of their ages, A said that the product of their ages was 750. B replied, that if his age were increased 7 years, and A's were lessened 2 years, their product would be 851. Required their ages.

Ans. A's 25 and B's 30 years.

23. John Smith's garden is a rectangle, and contains 15,000 square yards; and he, being a man of taste, has surrounded it with a walk 7 yards wide, the contents of which are 3696 square yards. Required the length and breadth of the garden.

Ans. Length 150, breadth 100 yards.

24. A gentleman purchased a farm for $5600, but if his farm had contained 10 acres more it would have cost him $10 less per acre. Of how many acres did his farm consist?

Ans. 70 acres.

25. A man purchased a farm in the form of a rectangle, whose length was four times its breadth. It cost $\frac{1}{4}$ as many

dollars per acre as the field was rods in length, and the number of dollars paid for the farm was four times the number of rods round it. Required the price of the farm, and its length and breadth.

Ans. Price $1600. Length 160 rods, breadth 40 rods.

26. Two men, A and B, set out from the same place at the same time to travel to Boston, it being 39 miles distant. A travelled $\frac{1}{4}$ of a mile an hour faster than B, and arrived at Boston an hour sooner. Required the rates of travelling.

Ans. A $3\frac{1}{4}$ and B 3 miles per hour.

27. What two numbcrs are those whose difference multiplied by the less produces 42, and by their sum 133?

Ans. 13 and 6.

28. A certain company agreed to build a vessel for $6300; but, two of their number having died, those that survived had each to advance $200 more than they otherwise would have done. Of how many persons did the company at first consist?

Ans. 9 persons.

29. I have a rectangular field of corn, which consists of 6250 hills, but the number of hills in the length exceeds the number in the breadth by 75. Of how many hills does the length and breadth consist? *Ans.* 125 hills the length, 50 the breadth.

30. A man bought 10 ducks and 12 turkeys for $22.50. He bought 4 more ducks for $6 than turkeys for $5. What was the price of each?

Ans. The price of a duck was 75 cents, and of a turkey $1.25.

31. What number is that to which if 24 be added, and the square root of the sum extracted, this root shall be less than the original quantity by 18? *Ans.* 25.

32. A has two gardens, each of which is an exact square. They contain 208 square rods. It requires 80 rods of fence to enclose both gardens. Required the contents of each.

Ans. 144 square rods; 64 square rods.

33. A has two square gardens, and it requires 80 rods of fence to enclose them. The larger contains 80 square rods

more than the other. How many square rods do both gardens contain? *Ans.* 208 square rods.

34. A has two square gardens, and the side of the one exceeds that of the other by 4 rods, and the contents of both are 208 square rods. How many square rods does the larger garden contain more than the smaller? *Ans.* 80 square rods.

35. I have two blocks of marble which are exact cubes, and whose united lengths are 20 inches, and they contain 2240 cubic inches. Required the surface of each.

Ans. Larger, 864 inches; smaller, 384 inches.

36. A merchant sold a bale of cloth for $75, and gained as much per cent. as the cloth cost him. What was the price of the cloth? *Ans.* $50.

37. There are two numbers whose difference is 12, and whose sum multiplied by the greater is 560. What are those numbers?

Ans. 20 and 8.

38. The plate of a looking-glass is 36 inches by 12 inches. It is to be framed with a frame all parts of which are of equal width, whose area is 448 square inches. What is the width of the frame? *Ans.* 4 inches.

39. Divide 100 into two such parts that the sum of their square roots shall be 14. *Ans.* 64 and 36.

40. A square court-yard has a rectangular gravel-walk around it. The side of the court wants one yard of being six times the breadth of the gravel-walk, and the number of square yards in the walk exceeds the number of yards in the perimeter of the court by 340. What is the area of the court and width of the walk?

Ans. Area of the court, 529 square yards; width of the walk, 4 yards.

41. A merchant bought 54 bushels of wheat, and a certain quantity of barley. For the former he gave half as many shillings per bushel as there were bushels of barley, and for the latter 4 shillings per bushel less. He sold the mixture at 10

shillings per bushel, and lost £28 16*s*. by his bargain. What was the price of the barley?

Ans. 36 bushels of barley, at 14 shillings per bushel.

42. I have $165\frac{1}{2}$ square feet of plank, 3 inches in thickness, with which I intend to make a cubical box. Required its contents in cubic feet. *Ans.* 125 cubic feet.

43. I have a small globule of glass, one inch in diameter. How large a sphere may be made of it, if the glass is to be only $\frac{1}{20}$ of an inch in thickness, taking it for *granted* that all spheres are to each other as the cubes of their diameters?

Ans. Inside diameter, 1.775+ inches; whole diameter, 1.875+ inches.

44. John Smith has two cubical boxes, whose united lengths in the clear are 20 inches, and their solid contents are 2240 cubic inches. What is the difference of their contents?

Ans. 1216 cubic inches.

45. I have two house-lots, which contain 6100 square feet, and the larger contains 1100 square feet more than the less. Required their dimensions. *Ans.* 50 and 60 feet square.

46. Two men, A and B, bought a farm of 200 acres, for which they paid $200 each. On dividing the land, A says to B, If you will let me have my part in the situation which I shall choose, you shall have so much more land than I that mine shall cost 75 cents per acre more than yours. B accepted the proposal. How much land did each have, and what was the price of each per acre?

Ans. A had 81.866 acres, at $2.443+; B had 118.133+ acres, at $1.693+.

47. A and B engaged to reap a field for 90 shillings. A could reap it in 9 days, and they promised to complete it in 5 days. They found, however, that they were obliged to call in C, an inferior workman, to assist them the last two days, in consequence of which B received 3*s*. 9*d*. less than he otherwise would have done. In what time could B and C each reap the field?

Ans. B could reap the field in 15 days, and C in 18 days.

SECTION XVIII.

CUBIC AND HIGHER EQUATIONS.

Art. 226. A Cubic Equation is one in which the highest power of the unknown quantity is the third power.

As, $x^3-ax^2+bx=c.$

227. A Biquadratic is an equation in which the highest power of the unknown quantity is the fourth power.

As, $x^4-ax^3+bx^2-cx=d.$

228. An equation of the fifth degree is one in which the highest power of the unknown quantity is the fifth power.

As, $x^5-ax^4+bx^3-cx^2+dx=e.$

And so on, for all other higher powers.

There are many particular and very prolix rules given for the solution of the above-mentioned equations; but they all may be readily solved by the following easy

Rule. 1. *Find, by trial, two quantities as near the true root as convenient, and substitute them separately, in the given equation, instead of the unknown quantity, and find how much the terms collected together, according to their signs + or —, differ from the known members of the equation, noting whether these errors are in excess or deficiency.*

2. *Multiply the difference of the two quantities found, or taken by trial, by either of the errors, and divide the product by the difference of the errors when they are alike, but by their sum when they are unlike. Or, we may say, as the difference or sum of the errors is to the difference of the two assumed quantities, so is either error to the correction of its supposed quantity.*

3. *Add the quotient last found to the quantity belonging to that error when its supposed quantity is too little, but subtract it when too great, and the result will give the true root* nearly.

4. *Take this root, and the nearer of the two former, or any*

other that may be found nearer; and, by proceeding in like manner as above, a root will be obtained nearer than before. Proceeding in the same manner, we may obtain the answer to any degree of exactness required.

Note 1.—It is best always to employ two assumed quantities, that shall differ from each other only by unity in the last figure on the right, because then the difference, or multiplier, is only 1. It is also best to use always the less error in the above operation.

Note 2.—It will be convenient, also, to begin with a single figure at first, trying several single figures, till there be found the two nearest the truth, the one too little, and the other too great; and, in working with them, find only one more figure. Then substitute this corrected result in the equation for the unknown letter; and, if the result prove too little, substitute also the number next greater for the second supposition; but, if the former prove too great, then take the next less number for the second supposition; and, working with the second pair of errors, continue the quotient only so far as to have the corrected number to four places of figures. Then repeat the same process again with this last corrected number, and the next greater or less, as the case may require, carrying the third corrected number to eight figures, because each new operation commonly doubles the number of true figures. Proceed in this manner to any extent that may be wanted.

EXAMPLES.

1. Find the root of the cubic equation $x^3+x^2+x=100$.

We see that x lies between 4 and 5. We assume, therefore, 4 and 5 as the two values of x.

FIRST SUPPOSITION.				SECOND SUPPOSITION.
4	=	x	=	5
16	=	x^2	=	25
64	=	x^3	=	125
84		sums		155
100		but should be		100
-16		errors		$+55$

Sum of the errors, $55+16=71$.

Then, $71 : 1 :: 16 : .2$.

Hence, $x=4+.2=4.2$ nearly.

Again, let $x=4.2$ and 4.3.

FIRST SUPPOSITION.		SECOND SUPPOSITION.
4.2	x	4.3
17.64	x^2	18.49
74.088	x^3	79.507
95.928	sums	102.297
100		100
−4.072		+2.297

Sum of the errors, $4.072+2.297=6.369$.

As $6.369 : .1 :: 2.297 : 0.036$.

Hence $x=4.3-.036=4.264$ nearly.

Again, let $x=4.264$ and 4.265.

FIRST SUPPOSITION.		SECOND SUPPOSITION.
4.264	x	4.265
18.181696	x^2	18.190225
77.526752	x^3	77.581310
99.972448		100.036535
100		100
0.027552		0.036535

Sum of the errors, $.027552+.036535=.064087$.

As $.064087 : .001 :: .027552 : 0.0004299$.

Hence, $x=4.264+.0004299=4.2644299$ nearly.

2. Find the root of the equation $x^3-15x^2+63x=50$.

Here it is evident that the root is more than 1. We then assume the two values of x to be 1.0 and 1.1.

Then	63.0	=	$63x$	=	69.3
	−15	=	$-15x^2$		−18.15
	1	=	x^3		1.331
	49		sums		52.481
	50				50
	−1		errors		+2.481

Sum of the errors, $1+2.481=3.481$.

As 3.481 : .1 : : 1 : .03
Add 1.00

Hence x = 1.03 nearly.

Again, let $x=1.03$ and 1.02.

Then	64.89	$63x$	64.26
	-15.9135	$-15x^2$	-15.6060
	1.092727	x^3	1.061208
	50.069227	sums	49.715208
	50		50
	$+.069227$	errors	$-.284792$
	.284792		
	.354019		

.354019 : .01 : : .069227 : .0019555.

Hence $x=1.03-.0019555=1.02804$ nearly.

3. Find the value of x in the equation $x^3+10x^2+5x=260$.
Ans. $x=4.1179857$.

4. Find the value of x in the equation $x^3-2x=50$.
Ans. $x=3.8648854$.

5. Find the value of x in the equation $x^4-3x^2-75x=10000$.
Ans. $x=10.2609$.

6. Find the value of x in the equation $x^5+2x^4+3x^3+4x^2+5x=54321$. *Ans.* $x=8.414455$.

7. I have a cubical block of marble, and if the superficial contents were added to its solid contents, the sum would be 432 feet. What is the length of the block? *Ans.* 6 feet.

8. Five times the cube of a certain number exceeds ten times its square by 45. Required the number. *Ans.* 3.

9. The fourth power of a certain number exceeds ten times its square by 375. Required the number. *Ans.* 5.

SECTION XIX.

RATIOS.

ART. **229.** Ratio is the relation which one quantity bears to another of a similar kind, with respect to its magnitude.

230. The magnitude or value of a ratio is estimated by stating how often one quantity or number contains or is contained in another. Thus, in comparing 16 with 2, we observe that it has a certain relative magnitude with respect to 2, which it contains 8 times; and, if we compare 16 with 4, we observe that it has a different relative magnitude, for it contains 4 only 4 times. Hence, 16 is less relatively, when compared with 4, than it is when compared with 2.

231. The general method of expressing the ratio which one quantity or number bears to another is by placing two points between them. Thus,

The ratio of 12 to 4 is expressed by $12 : 4$.
" 19 to 9 " " by $19 : 9$.
" a to b " " by $a : b$.

232. The first term of a ratio is called the *Antecedent*, and the last term the *Consequent*. The antecedents in the preceding ratios are, therefore, 12, 19, and a, and the consequents 4, 9, and b.

233. Ratios may, therefore, be represented in the form of fractions, by making the antecedents the numerators, and the consequents the denominators; thus,

$$\frac{12}{4},\ \frac{19}{9},\ \text{and}\ \frac{a}{b},$$

express the ratios of 12 to 4, of 19 to 9, and of a to b.

234. A ratio is said to be of equality when the antecedent is equal to the consequent.

Thus the ratio of $12 : 12$, or of $a : a$, is a ratio of equality.

235. A ratio is of greater inequality when the antecedent is greater than the consequent. Thus,

The ratio of $a+b : a$, or of $12 : 6$, is a ratio of greater inequality.

236. A ratio of less inequality is when the antecedent is less than the consequent. Thus,

The ratio of $a : a+b$, or of $6 : 12$, is a ratio of less inequality.

NOTE.—It is evident that the ratio of equality may always be represented by unity.

COMPARISON BY RATIOS.

237. If the terms of a ratio are both multiplied or both divided by the same quantity, the value of the ratio is not altered.

The ratio of $a : b$ is expressed by the fraction $\frac{a}{b}$. Let both terms of this fraction be multiplied by n, and it becomes $\frac{na}{nb}$. The ratio of $4 : 3$ is expressed by the fraction $\frac{4}{3}$; and, if the terms of this fraction be multiplied by 3, it becomes $\frac{12}{9}=\frac{4}{3}$. Now, since the value of a fraction is not altered by multiplying both the numerator and denominator by the same quantity, $\frac{a}{b}=\frac{na}{nb}$, or the ratio $a : b$ is the same as the ratio $na : nb$, and the ratio of $12 : 9$ is the same as $4 : 3$. Thus the ratio of $16 : 12$, both terms being divided by 4, is the same as $4 : 3$.

The ratio of $5 : 7$, both terms being multiplied by 3, is the same as the ratio of $15 : 21$. And the ratio of $a^2 : ab$, by dividing by a, is the same as the ratio of $a : b$.

238. Ratios are compared together by reducing the fractions which represent them to a common denominator.

Thus the ratios of $7 : 9$ and $10 : 13$ are represented by the fractions $\frac{7}{9}$ and $\frac{10}{13}$, which are equivalent to $\frac{91}{117}$ and $\frac{90}{117}$; and since $\frac{91}{117}$ is greater than $\frac{90}{117}$, we infer that the ratio of $7 : 9$ is greater than $10 : 13$.

239. When the antecedents or consequents are the same in two or more ratios, we immediately compare those ratios together by expressing them in a fractional form. Thus, since $\frac{17}{5}$ is greater than $\frac{17}{9}$, the ratio of $17 : 5$ is greater than $17 : 9$; and, since $\frac{a}{a+b}$ is less than $\frac{a}{b}$, the ratio of $a : a+b$ is less than $a : b$.

240. A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both terms.

Let $\frac{a}{b}$ represent any ratio, and add n to each of the terms then these two ratios will be $\frac{a}{b}$ and $\frac{a+n}{b+n}$, which are equivalent to $\frac{ab+an}{b(b+n)}$ and $\frac{ab+bn}{b(b+n)}$. Now, if a be greater than b, $\frac{a}{b}$ is a ratio of greater inequality, and $\frac{ab+an}{b(b+n)}$ is greater than $\frac{ab+bn}{b(b+n)}$, therefore $\frac{a}{b}$ is diminished by adding n to each of the terms. But, if a be less than b, then $\frac{a}{b}$ is a ratio of less inequality, and $\frac{ab+an}{b(b+n)}$ is less than $\frac{ab+bn}{b(b+n)}$; therefore, $\frac{a}{b}$ is increased by the addition of n to both terms.

COMPOUND RATIOS.

241. Ratios are compounded by multiplying their antecedents together to form a new antecedent, and their consequents *to* form a new consequent. The resulting ratio is called the *sum* of the compounding ratios.

Thus, the ratio of $a : b$ is compounded with the ratio of $c : d$ by multiplying the antecedents a and c together for a new antecedent, and the consequents b and d together for a new consequent, and the resulting ratio $ac : bd$ is the sum of the compounding ratios $a : b$ and $c : d$.

If the ratios $4 : 7$, $6 : 11$, and $7 : 9$ are compounded together, the resulting ratio is $4\times6\times7 : 7\times11\times9$, or $168 : 693$, which, reduced to its lowest terms by dividing both terms by 21, becomes the ratio $8 : 33$.

242. When any ratio, $a : b$, is compounded with itself twice, thrice, or any number of times, denoted by n, then the resulting ratios are $a^2 : b^2$, $a^3 : b^3$, $a^4 : b^4$, &c., and are called the duplicate, triplicate, quadruplicate, &c., ratios of the primitive.

As the indices or exponents, 2, 3, and n, express the number of times the ratio of $a : b$ is compounded of itself, they are called the *measures* of these ratios.

Since the index may be any quantity, either integral or fractional, let the fraction be $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{m}$, &c.; then,

The ratio of $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is the square root of the ratio of $a : b$.
" " $a^{\frac{1}{3}} : b^{\frac{1}{3}}$ is the cube root of " " of $a : b$.
" " $a^{\frac{1}{4}} : b^{\frac{1}{4}}$ is the fourth root of " " of $a : b$.
" " $a^{\frac{1}{m}} : b^{\frac{1}{m}}$ is the $\frac{1}{m\text{th}}$ root of " " of $a : b$.

243. The ratios of $a^{\frac{1}{2}} : b^{\frac{1}{2}}$, $a^{\frac{1}{3}} : b^{\frac{1}{3}}$, $a^{\frac{1}{4}} : b^{\frac{1}{4}}$, &c., are also called the subduplicate, subtriplicate, subquadruplicate, &c., ratios of a to b.

PROPORTION.

244. Proportion consists in the equality of ratios.

Thus, if the ratio of $a : b$ is equal to that of $c : d$, or $\frac{a}{b}=\frac{c}{d}$, then a, b, c, d, are said to be *proportional.* The numbers 3, 12, 4, 16, are proportionals, for $\frac{3}{12}=\frac{1}{4}$, and $\frac{4}{16}=\frac{1}{4}$.

This equality of ratios is expressed by writing the four quantities thus, $a : b :: c : d$, and is read, a is to b as c to d.

245. In algebraic investigations the quantities are generally expressed like fractions, thus $\frac{a}{b}=\frac{c}{d}$.

In the proportion $a : b :: c \cdot d$, or $\frac{a}{b}=\frac{c}{d}$, a and d are the extremes, and b and c the means. The first term is also called the first antecedent, and the second the first consequent, the third term the second antecedent, and the fourth term the second consequent.

246. If in a series of proportional quantities each consequent is identical with the next antecedent, these quantities are said to be in continued proportion. Thus, $a : b :: b : c :: c : d :: d : e :: e : f$, &c., the quantities a, b, c, d, e, f, &c., are said to be in continued proportion.

247. When the second and third terms are identical, as in the proportion $a : b :: b : c$, then b is said to be a mean proportional between the extremes a and c, and c is called the third proportional to a and b.

248. The product of any number of ratios, of which the consequent of each ratio is the antecedent of the succeeding one, is the ratio of the first antecedent to the last consequent.

Let the ratios be $a : b$, $b : c$, $c : d$, $d : e$, $e : f$, then the resulting ratio is $a\times b\times c\times d\times e : b\times c\times d\times e\times f$, or the ratio of $abcde : bcdef$, which, reduced to its least terms by cancelling the same letters in each term, becomes $a : f$, or the first antecedent and the last consequent.

Again; let the ratios be $2 : 3$, $3 : 4$, $4 : 5$, $5 : 7$, $7 : 10$, then the resulting ratio is,

$2\times3\times4\times5\times7 : 3\times4\times5\times7\times10$, or $840 : 4200$,

which reduced is, $7 : 35$, or $1 : 5$.

249. Any ratio compounded with a ratio of greater inequality is increased, and compounded with a ratio of less inequality is diminished.

Let $a+b : a$ represent the ratio of greater inequality, and $a : a+b$ of less inequality. Then the ratio of $a+b : a$, compounded with that of $c : d$, gives $ac+bc : ad$, which is evidently greater than the ratio of $c : d$; and the ratio of $a : a+b$, compounded with that of $c : d$, gives $ac : ad+bd$, which is evidently less than the ratio of $c : d$.

Hence the ratio of $c : d$ is increased by compounding it with the ratio of $a+b : a$, and diminished by compounding it with the ratio of $a : a+b$.

APPROXIMATION OF RATIOS.

250. The ratio of the powers or roots of two quantities whose difference is small with respect to themselves is found very nearly by multiplying that difference by the index or exponent of the power or root.

PROPOSITIONS.

PROPOSITION I. If four quantities are proportional, the product of the extremes is equal to the product of the means, and conversely.

Let $a : b :: c : d$, or $\frac{a}{b}=\frac{c}{d}$.

Multiplying both by bd, we obtain $ad=bc$.

Conversely. If the product of any two quantities is equal to the product of any other two, these four quantities are proportional, the factors of either of the products being made the extremes, and the factors of the other the means.

Let $ad=bc$, dividing both by bd, we obtain $\frac{a}{b}=\frac{c}{d}$, or $\frac{c}{d}=\frac{a}{b}$; whence $a : b :: c : d$, or $c : d :: a : b$.

PROP. II. If three quantities are in continued proportion, the product of the extremes is equal to the square of the mean, and conversely.

Let $a : b :: b : c$; $a\times c=b\times b$, or $ac=b^2$.

Conversely. If the product of any two quantities is equal to the square of a third, the third is a mean proportional between the other two.

Let $ac=b^2$; and, dividing both by bc, we obtain $\frac{a}{b}=\frac{b}{c}$, or $a : b :: b : c$.

Prop. III. Of four proportionals, any three being given, the fourth may be found.

Let $\qquad a : b :: c : d$; then $ad=bc$.

Hence, $\qquad a=\frac{bc}{d}$; $b=\frac{ad}{c}$; $c=\frac{ad}{b}$; $d=\frac{bc}{a}$.

Hence, of three proportionals, any two being given, the third may be found; for $ad=b^2$, therefore $b=\sqrt{ad}$, $a=\frac{b^2}{d}$, and $d=\frac{b^2}{a}$.

Prop. IV. Quantities which have the same ratio to the same quantity are equal to one another, and conversely.

Let $a : b :: c : b$, then $\frac{a}{b}=\frac{c}{b}$; and, multiplying each by b, we obtain $a=c$.

Conversely. Quantities which are equal to one another have the same ratio to the same quantity.

Let $a=c$, and let b be a third quantity; then, dividing both by b, we obtain

$$\frac{a}{b}=\frac{c}{b}, \text{ therefore } a : b :: c : b.$$

Prop. V. Ratios that are equal to the same ratio are equal to one another.

Let $a : b :: e : f$, and $c : d :: e : f$; then, also, $a : b :: c : d$.

Since $\frac{a}{b}=\frac{e}{f}$, and $\frac{c}{d}=\frac{e}{f}$, it is evident $\frac{a}{b}=\frac{c}{d}$, and therefore $a : b :: c : d$.

Or let $\qquad 2 : 4 :: 8 : 16$, and $3 : 6 :: 8 : 16$.

Then $\qquad 2 : 4 :: 3 : 6$; for $\frac{2}{4}=\frac{1}{2}$, and $\frac{3}{6}=\frac{1}{2}$.

Prop. VI. If four quantities are proportionals, they will also be proportionals when taken *inversely;* that is, the second will have the same ratio to the first that the fourth has to the third.

Let $a : b :: c : d$, then $b : a :: d : c$.

Since by Prop. I. $bc = ad$,

And, dividing by ac, we obtain $\frac{b}{a} = \frac{d}{c}$,

Hence, $b : a :: d : c$.

Prop. VII. If four quantities are proportionals, they will also be proportionals when taken alternately; that is, the first will have the same ratio to the third that the second has to the fourth.

Let $a : b :: c : d$; then, also, $a : c :: b : d$;

As $\frac{a}{b} = \frac{c}{d}$, if we multiply each quantity by $\frac{b}{c}$, we obtain $\frac{ab}{bc} = \frac{cb}{cd}$; which, reduced, is $\frac{a}{c} = \frac{b}{d}$, therefore $a : c :: b : d$.

This may be illustrated by numbers; thus,

Let $2 : 4 :: 3 : 6$, then $2 : 3 :: 4 : 6$;

As $\frac{2}{4} = \frac{3}{6}$, if we multiply each side of the equation by $\frac{4}{3}$, the result will be $\frac{2}{4} \times \frac{4}{3} = \frac{3}{6} \times \frac{4}{3}$; $\frac{8}{12} = \frac{12}{18}$; $\frac{2}{3} = \frac{4}{6}$; therefore $2 : 3 :: 4 : 6$.

Prop. VIII. If four quantities are proportionals, they will also be proportionals when taken jointly; that is, the sum of the first and second will have the same ratio to the second that the sum of the third and fourth has to the fourth.

Let $a : b :: c : d$, then $a + b : b :: c + d : d$.

Since $\frac{a}{b} = \frac{c}{d}$, we add 1 to each quantity, and obtain $\frac{a}{b} + 1 = \frac{c}{d} + 1$, or $\frac{a+b}{b} = \frac{c+d}{d}$, therefore $a + b : b :: c + d : d$.

This, also, may be made evident by taking numbers; thus,

Let $2 : 4 :: 3 : 6$, then $2 + 4 : 4 :: 3 + 6 : 6$.

Since $\frac{2}{4}=\frac{3}{6}$, we add 1 to each number, and obtain

$$\frac{2}{4}+1=\frac{3}{6}+1, \text{ or } \frac{2+4}{4}=\frac{3+6}{6}.$$

Therefore, $2+4 : 4 :: 3+6 : 6$.

Prop. IX. If four quantities are proportionals, they will also be proportionals by *separation;* that is, the difference between the first and second will have the same ratio to the second that the difference between the third and fourth has to the fourth.

Let $a : b :: c : d$, then $a-b : b :: c-d : d$.

Since $\frac{a}{b}=\frac{c}{d}$, we will subtract 1 from each quantity, and we obtain $\frac{a}{b}-1=\frac{c}{d}-1$, or $\frac{a-b}{b} : \frac{c-d}{d}$.

Therefore, $a-b : b :: c-d : d$.

This demonstration may be illustrated by numbers; thus,

Let $4 : 2 :: 6 : 3$, then $4-2 : 2 :: 6-3 : 3$.

Since $\frac{4}{2}=\frac{6}{3}$, we subtract 1 from each term, and we have

$$\frac{4}{2}-1=\frac{6}{3}-1, \text{ or } \frac{4-2}{2}=\frac{6-3}{3}.$$

Therefore, $4-2 : 2 :: 6-3 : 3$.

Prop. X. If four quantities are proportionals, they will also be proportionals by *conversion;* that is, the first term will have the same ratio to the sum or difference of the first and second, that the third has to the sum or difference of the third and fourth.

Let $a : b :: c : d$; then $a : a\pm b : c :: c\pm d$. Since $\frac{a}{b}=\frac{c}{d}$, and, by Prop. VIII. and IX., $\frac{a\pm b}{b}=\frac{c\pm d}{d}$, invert these fractions, and we have $\frac{b}{a\pm b}=\frac{d}{c\pm d}$; and, by multiplying the one by $\frac{a}{b}$, and the other by its equal $\frac{c}{d}$, we obtain $\frac{b}{a\pm b}\times\frac{a}{b}=\frac{d}{c\pm d}\times\frac{c}{d}$, or $\frac{a}{a\pm b}=\frac{c}{c\pm d}$, therefore $a : a\pm b :: c : c\pm d$.

Let the pupil prove this by numbers.

Prop. XI. If four quantities are proportionals, the sum of the first and second has the same ratio to their difference that the sum of the third and fourth has to their difference.

Let $a : b :: c : d$; then, also, $a+b : a-b :: c+d : c-d$.

For, by Prop. VIII. and IX. by alternation, $a+b : c+d :: b : d$; and $a-b : c-d :: b : d$; hence, by Prop. V., $a+b : c+d :: a-b : c-d$, and, by alternation, $a+b : a-b :: c+d : c-d$.

This is illustrated by numbers, thus; let $8 : 6 :: 12 : 9$; then $8+6 : 8-6 :: 12+9 : 12-9$.

For taking Prop. VIII. and IX. by alternation, $8+6 : 12+9 :: 6 : 9$; and by Prop. V., $8+6 : 12+9 :: 8-6 : 12-9$; therefore, by alternation, $8+6 : 8-6 :: 12+9 : 12-9$.

Prop. XII. In any number of proportionals, any antecedent has the same ratio to its consequent that the sum of all the antecedents has to the sum of all the consequents.

Let $a : b :: c : d :: e : f :: g : h$; then, also, $a : b :: a+c+e+g : b+d+f+h$.

Since $ab=ba$, $ad=bc$, $af=be$, $ah=bg$, we have

$$a(b+d+f+h)=b(a+c+e+g);$$

Whence, $$\frac{a}{b}=\frac{a+c+e+g}{b+d+f+h};$$

Therefore, $a : b :: a+c+e+g : b+d+f+h$.

In like manner it may be shown that $c : d :: a+c+e+g$ $b+d+f+h$.

This proposition may be illustrated by numbers, thus,

Let $2 : 3 :: 4 : 6 :: 8 : 12 :: 14 : 21$;

Then $2 : 3 :: 2+4+8+14 : 3+6+12+21 = 2 : 3 :: 28 : 42$.

Prop. XIII. In two or more sets of proportionals, the product of the correspondent terms are also proportionals.

$$\text{Let } \left.\begin{array}{l} a : b :: c : d, \\ e : f :: g : h, \\ i : k :: l : m, \end{array}\right\} \text{Then, also, } aei : bfk :: cgl : dhm.$$

DEMONSTRATION.

Since $\frac{a}{b}=\frac{c}{d}, \frac{e}{f}=\frac{g}{h}, \frac{i}{k}=\frac{l}{m}; \frac{a\times e\times i}{b\times f\times k}=\frac{c\times g\times l}{d\times h\times m};$

Whence $\frac{aei}{bfk}=\frac{cgl}{dhm}$, therefore, $aei : bfk : cgl :: dhm$.

Q. E. D.

ILLUSTRATION BY NUMBERS.

$$\begin{array}{ll} \text{Let} & 2 : 3 :: 4 : 6 \\ & 4 : 5 :: 8 : 10 \\ & 6 : 7 :: 12 : 14 \end{array}$$

Then $2\times4\times6 : 3\times5\times7 :: 4\times8\times12 : 6\times10\times14$.

Whence, $48 : 105 :: 384 : 840$.

Prop. XIV. If there are any number of quantities more than two, and as many others, which, taken two and two in order, are proportionals, then, by equality, are the extreme terms in the former series proportional to the extreme terms in the latter.

Let a, b, c, d, be any number of quantities, and let e, f, g, h, be as many others.

$$\text{Let } \left.\begin{array}{l} a : b :: e : f, \\ b : c :: f : g, \\ c : d :: g : h, \end{array}\right\} \text{Then, also, } a : d :: e : h.$$

DEMONSTRATION.

Since $\frac{a}{b}=\frac{e}{f}, \frac{b}{c}=\frac{f}{g}$, and $\frac{c}{d}=\frac{g}{h}$, we obtain, by multiplying the alternate fractions together, $\frac{abc}{bcd}=\frac{efg}{fgh}$, or $\frac{a}{d}=\frac{e}{h}$; therefore, $a : d :: e : h$.

ILLUSTRATION BY NUMBERS.

$$\text{Let } \left.\begin{array}{l} 2 : 3 :: 4 : 6 \\ 3 : 4 :: 6 : 8 \\ 4 : 12 :: 8 : 24 \end{array}\right\} \text{Then } 2 : 12 :: 4 : 24.$$

By multiplying the alternate fractions, we have

$$2\times3\times4 : 3\times4\times12 :: 4\times6\times8 : 6\times8\times24.$$

Whence, $24 : 144 :: 192 : 1152$, or $2 : 12 :: 4 : 24$.

Prop. XV. If there are any number of quantities more than two, and as many others, which, taken two and two, in cross order, are proportionals, then inversely, by equality, are the extreme terms in the first set proportional to the extreme terms in the second.

Let a, b, c, d, be any number of terms, and e, f, g, h, as many others, and

$$\text{Let}\quad \left.\begin{array}{l} a : b :: g : h \\ b : c :: f : g \\ c : d :: e : f \end{array}\right\}\text{Then, also, } a : d :: e : h.$$

DEMONSTRATION.

Since $\frac{a}{b}=\frac{g}{h}$, $\frac{b}{c}=\frac{f}{g}$, and $\frac{c}{d}=\frac{e}{f}$, by multiplying the alternate fractions together, we obtain

$$\frac{abc}{bcd}=\frac{gfe}{hgf}, \text{ or } \frac{a}{d}=\frac{e}{h},$$

Therefore, $a : d :: e : h$.

ILLUSTRATION BY NUMBERS.

$$\text{Let}\quad \left.\begin{array}{l} 2 : 3 :: 8 : 12 \\ 3 : 4 :: 6 : 8 \\ 4 : 3 :: 8 : 6 \end{array}\right\}\text{Then, } 2 : 3 :: 8 : 12.$$

$$2\times3\times4 : 3\times4\times3 :: 8\times6\times8 : 12\times8\times6.$$

Whence, $24 : 36 :: 384 : 576.$

By dividing the first two terms by 12, and the last two by 48, we obtain $2 : 3 :: 8 : 12.$

Prop. XVI. When four quantities are proportionals, if the first and second are multiplied or divided by the same quantity, and also the third and fourth by the same quantity, the resulting quantities will be proportionals.

Let $a : b :: c : d$; then, also, $ma : mb :: nc : nd$.

DEMONSTRATION.

Since $\frac{a}{b}=\frac{c}{d}$, we multiply both terms of the first by m, and both terms of the last by n, and we obtain $\frac{ma}{mb}=\frac{nc}{nd}$;

Therefore, $ma : mb :: nc : nd$,
where m and n may be any quantities, either integral or fractional.

ILLUSTRATION BY NUMBERS.

Let 2 : 4 : : 3 : 6. Now, if we multiply the first two numbers by 7, and the last two numbers by 9, their products will be proportionals. Thus,

$$2\times7 : 4\times7 :: 3\times9 : 6\times9 = 14 : 28 :: 27 : 54;$$

and if any other numbers were taken instead of 7 and 9, the products would be proportionals.

PROP. XVII. When four quantities are proportionals, if the first and third are multiplied or divided by the same quantity, and also the second and fourth by the same quantity, the resulting quantities will be proportionals.

Let $a : b :: c : d$, then, also, $ma : nb :: mc : nd$.

DEMONSTRATION.

Since $\frac{a}{b}=\frac{c}{d}$, multiply both these quantities by $\frac{m}{n}$, and we obtain $\frac{ma}{nb}=\frac{mc}{nd}$, therefore, $ma : nb :: mc : nd$, where m and n may be any quantities, either integral or fractional.

ILLUSTRATION BY NUMBERS.

Let 12 : 4 : : 18 : 6, and we will multiply the first and third by 2, and the second and fourth terms by 4.

Thus, $12\times2 : 4\times4 :: 18\times2 : 6\times4 = 24 : 16 :: 36 : 24$.

It is evident these terms are proportionals;

For $$\frac{24}{16}=\frac{36}{24}, \text{ or } \frac{12}{8}=\frac{12}{8}.$$

And if we divide the first and third terms by 3, and the second and fourth terms by 2, their quotients will be proportionals.

Thus, $12\div3 : 4\div2 :: 18\div3 : 6\div2$.

Or $4 : 2 :: 6 : 3$.

Whence, $\frac{4}{2}=\frac{6}{3}$.

If any other numbers be taken for multiplying or dividing, the result will be the same.

PROP. XVIII. If four quantities are proportionals, the like powers or roots of these quantities are also proportionals.

Let $a : b :: c : d$; then, also, $a^m : b^m :: c^m : d^m$.

Since $\frac{a}{b}=\frac{c}{d}$, raise each of these fractions to the power expressed by m; then $\left(\frac{a}{b}\right)^m=\left(\frac{c}{d}\right)^m$, or $\frac{a^m}{b^m}=\frac{c^m}{d^m}$, therefore, $a^m : b^m :: c^m : d^m$, where m may be any quantity, either integral or fractional.

ILLUSTRATION.

Let $2 : 3 :: 4 : 6$, then $2^3 : 3^3 :: 4^3 : 6^3$. If we raise each of these terms to the third power, the result will be

$2\times2\times2=8 : 3\times3\times3=27 :: 4\times4\times4=64 : 6\times6\times6=216$.

That 8, 27, 64, and 216, are proportionals, is evident from the fact that $\frac{8}{27}=\frac{64}{216}$, and, being reduced to their lowest terms, $\frac{8}{27}=\frac{8}{27}$.

PROP. XIX. Of any number of quantities in continued proportion, the first has to the third the duplicate ratio, to the fourth the triplicate ratio, to the fifth the quadruplicate ratio, &c., of that which it has to the second, or of that which the second has to the third, &c.

Let $a : b :: b : c :: c : d :: d : e :: e : f ::$ &c. &c.

Then $a : c :: a^2 : b^2$, or in the duplicate ratio of $a : b$.

$a : d :: a^3 : b^3$, or in the triplicate ratio of $a : b$.

$a : e :: a^4 : b^4$, or in the quadruplicate ratio of $a : b$.

DEMONSTRATION.

1st. $a : b :: b : c$, or, by Prop. XVIII., $a^2 : b^2 :: b^2 : c^2$; but, by Prop. II., $b^2=ac$, therefore, $a^2 : b^2 :: ac : c^2$, or $a^2 : b^2 :: a : c$, hence $a : c :: a^2 : b^2$; also, $a^2 : ac :: b^2 : c^2$; therefore, $a : c :: b^2 : c^2$.

2d. $a : c :: a^2 : b^2$; but $c : d :: a : b$; therefore,

$$a : d :: a^3 : b^3 :: b^3 : c^3 :: c^3 : d^3.$$

3d. But $d : e :: a^3 : b^3$; therefore,

$$a : e :: a^4 : b^4 :: b^4 : c^4 :: c^4 : d^4 :: d^4 : e^4.$$

The above may be easily illustrated by numbers.

PROBLEMS FOR PROPORTION.

1. Divide 50 into two such parts that the greater, increased by 3, shall be to the less, diminished by 3, as 3 to 2.

Let $x =$ the greater number, and $50-x$ the less.

Then $x+3 : 50-x-3 :: 3 : 2$.

Multiplying extremes, $2x+6=150-3x-9$.

Transposing, $5x=135$.

Dividing, $x=27$, the greater.

And $50-27=23$, the less.

2. What number is that to which if 3, 8, 12, and 20, be severally added, their sums shall be proportional?

Let $x =$ the number.

Then, $x+3 : x+8 :: x+12 : x+20$.

Multiplying extremes, $x^2+23x+60=x^2+20x+96$.

Transposing, $23x-20x=96-60$.

Dividing, $x=12$. *Ans.*

VERIFICATION.

$$12+3 : 12+8 :: 12+12 : 12+20=15 : 20 :: 24 : 32.$$

3. If Mars, when in opposition to the sun, is 49,000,000 miles from the earth, and the quantity of matter in the earth is 11 times greater than that in Mars, at what distance from the earth, in a direction towards Mars, will a body remain at rest? See Art 218.

Let $x =$ the distance from the earth.

Then $49{,}000{,}000-x =$ the distance from Mars.

And let $a=49{,}000{,}000$.

Then, $x^2 : (a-x)^2 :: 1 : 11$.

Multiplying extremes, $11x^2=a^2-2ax+x^2$.

Transposing, $10x^2+2ax=a^2$.

Reducing, $x^2+\frac{ax}{5}=\frac{a^2}{10}$.

Completing the squares, $x^2+\frac{ax}{5}+\frac{a^2}{100}=\frac{a^2}{10}+\frac{a^2}{100}=\frac{11a^2}{100}$.

Evolving, $x+\frac{a}{10}=\frac{1}{10}\sqrt{11a^2}$.

Transposing, &c. $x=\frac{a}{10}\sqrt{11}-\frac{a}{10}$.

And, by supplying the value of a, we have

$$x=\frac{1}{10}\sqrt{\left(11(49{,}000{,}000)^2\right)}-\frac{49{,}000{,}000}{10}=11{,}351{,}430 \text{ miles.}$$

Ans.

4. There are two numbers which are to each other as 5 to 3; and, if 4 be added to the greater and 8 to the less, they will then be to each other as 6 to 5. What are the numbers?
Ans. 20 and 12.

5. Divide the number 60 into two such parts that their product shall be to the difference of their squares as 2 to 3.
Ans. 40 and 20.

6. I have two square house-lots, which, together, contain 208 square rods; and the area of the greater is to the area of the less as 9 to 4. How many more square rods are there in the greater than in the less? *Ans.* 80 square rods.

7. The product of two numbers is 12, and the difference of their cubes is to the cube of their difference as 13 to 4. What are the numbers? *Ans.* 2 and 6.

8. Divide the number 100 into two such parts that 6 times their product shall be to the sum of their squares as 24 to 17. What are those parts? *Ans.* 80 and 20.

9. There are two numbers, whose product is 35, and the difference of their squares is to the square of their difference as 6 to 1. What are the numbers? *Ans.* 7 and 5.

10. There are two numbers in the triplicate ratio of 4 to 1. whose mean proportional is 32. What are the numbers?

Ans. 256 and 4.

11. Divide 20 into two such numbers, that the quotient of the greater divided by the less shall be to the quotient of the less divided by the greater as 9 to 4. What are those numbers?

Ans. 12 and 8.

12. Divide 26 into three such parts, that the first shall have the same ratio to the second that the second has to the third, and that the first term shall be $\frac{1}{9}$ the third term.

Ans. 2, 6, and 18.

SECTION XX.

ARITHMETICAL PROGRESSION.

Art. 251. An Arithmetical Progression is a series of numbers or quantities, increasing or decreasing by a constant difference.

It is sometimes called *Progression by Difference.*

252. The constant difference is called the *Common Difference,* or *ratio* of the progression.

Ratio here used is an Arithmetical rate.

Thus, let there be the two following series.

(1) (2) (3) (4) (5) (6) (7) (8)
First series, 1, 4, 7, 10, 13, 16, 19, 22=92.
Second series, 30, 26, 22, 18, 14, 10, 6, 2=128.

253. The numbers which form the series are called the *terms* of the progression.

254. The first is called an *ascending series* of progression, where the first term is 1, the common difference 3, the number of terms 8, the last term 22, and the sum of the series 92.

255. The second is called a *descending series* of progression, where the first term is 30, the common difference —4, the number of terms 8, the last term 2, and the sum of the series 128.

256. The first and last terms of the progression are called *extremes*, and the other terms are the *means*.

257. The number of common differences in any number of terms is *one less* than the number of terms.

Hence, if there be 8 terms, the number of common differences will be 7, and the sum of the differences will be equal to the difference of the extremes.

We therefore infer, that if the difference of the extremes be added to the first term, the sum will be the last term; also, if the difference of the extremes be taken from the last term, the remainder will be the first term.

258. Also, if the sum of the common differences be divided by the number of common differences, the quotient will be the common difference.

To illustrate this, we will examine the following series:

(1)	(2)	(3)	(4)	(5)	(6)	(7)
2,	5,	8,	11,	14,	17,	20.

Here the first term is 2, the last term 20, the number of terms 7, and the common difference 3.

Now, if we had only the first term, number of terms, and common difference, to find the last term, we should have only to add the difference of the extremes to the first term.

The common difference is 3; and, as there are 7 terms, the number of common differences is 6. The difference of the extremes will, therefore, be $6 \times 3 = 18$, and the last term will be $2 + 18 = 20$.

Hence, having the first term, common difference, and number of terms given, to find the last term, we have the following

RULE. *Multiply the number of terms*, less one, *by the common difference, and to the product add the first term.*

Again, if we invert the terms, we have

(1)	(2)	(3)	(4)	(5)	(6)	(7)
20,	17,	14,	11,	8,	5,	2.

Here we have 20 for the first term, -3 for the common difference, and 7 for the number of terms, to find the last term.

$$6\times-3=-18;\ 20-18=2 \text{ the last term.}$$

The pupil will perceive that 18 is a negative term; and to add a negative term to a positive is to write their difference.

Again, we have given the extremes 2 and 20, and number of terms 7, to find the common difference.

Here the number of common differences is 6; for we have before shown that the number of common differences is always *one less* than the number of terms; therefore, $18\div6=3$, the common difference.

259. The principles of an arithmetical progression may be well illustrated by *literal terms.*

Let a be the first term of an ascending series, and d the common difference; then the second term will be $a+d$, and the the third term $a+2d$, and the series will be

(1)	(2)	(3)	(4)	(5)	(6)
a,	$a+d$,	$a+2d$,	$a+3d$,	$a+4d$,	$a+5d$.

If it be required to form a descending series, when the first term is a and the common difference $-d$, it will be thus:

(1)	(2)	(3)	(4)	(5)	(6)
a,	$a-d$,	$a-2d$,	$a-3d$,	$a-4d$,	$a-5d$.

260. It is evident that the last term in both series is equal to the first term with the common difference repeated as many times, *wanting one*, as there are terms in the series.

Hence, if n represent the number of terms, the following will be the formula to find L, the last term.

$$L=a+(n-1)d.$$

EXAMPLES.

1. If the first term be 7, the common difference 4, and the number of terms 20, required the last term.

$$L=a+(n-1)d=7+(20-1)4=83. \quad \textit{Ans.}$$

2. If the first term is 3, the common difference 5, required the 50th term.

$$L=a+(n-1)d=3+(50-1)5=248.\quad \textit{Ans.}$$

3. If the first term is 90, the common difference -7, required the 10th term.

$$L=a+(n-1)(-d)=90+(10-1)(-7)=27.\quad \textit{Ans.}$$

4. If the first term is $\frac{3}{4}$, the common difference $1\frac{1}{3}$, what is the 20th term?

$$L=a+(n-1)d=\tfrac{3}{4}+(20-1)1\tfrac{1}{3}=26\tfrac{1}{12}.\quad \textit{Ans.}$$

5. If the first term is 18, the common difference -4, what is the 10th term?

$$L=a+(n-1)(-d)=18+(10-1)(-4)=-18.\quad \textit{Ans.}$$

261. The formula for obtaining the first term, a, is obtained from the former by transposition.

Thus, if $L=a+(n-1)d$, then, by transposition,

$$a=L-(n-1)d.$$

6. If the last term is 25, the number of terms 6, and the common difference 2, required the first term.

$$a=L-(n-1)d=25-(6-1)2=15.\quad \textit{Ans.}$$

7. If the last term is 50, the common difference 6, the number of terms 10, required the first term.

$$a=L-(n-1)d=50-(10-1)6=-4.\quad \textit{Ans.}$$

8. If the last term is 27, the common difference $2\frac{1}{2}$, number of terms 10, required the first term.

$$a=L-(n-1)d=27-(10-1)2\tfrac{1}{2}=4\tfrac{1}{2}.\quad \textit{Ans.}$$

262. The formula for obtaining the common difference, d, is obtained from the first by transposition and division.

Thus, $L=a+(n-1)d.$

Then, by transposition, $L-a=(n-1)d.$

And by division, $\dfrac{L-a}{n-1}=d.$

Changing terms, $d=\dfrac{L-a}{n-1}.$

9. If the extremes are 6 and 30, and the number of terms 13, what is the common difference?

$$d=\frac{L-a}{n-1}=\frac{30-6}{13-1}=2. \quad \textit{Ans.}$$

10. If the extremes are $\frac{3}{4}$ and $15\frac{3}{4}$, and the number of terms 11, what is the common difference?

$$d=\frac{L-a}{n-1}=\frac{15\frac{3}{4}-\frac{3}{4}}{11-1}=1\tfrac{1}{2}. \quad \textit{Ans.}$$

263. The formula for obtaining the number of terms may be obtained from the first formula.

Thus, $L=a+(n-1)d.$

By transposition, $L-a=(n-1)d.$

By division, $\frac{L-a}{d}=n-1.$

By transposition, $\frac{L-a}{d}+1=n.$

Changing terms, $n=\frac{L-a}{d}+1.$

11. If the extremes are 3 and 39, and the common difference 2, what is the number of terms?

$$n=\frac{L-a}{d}+1=\frac{39-3}{2}+1=19. \quad \textit{Ans.}$$

12. If the first term is 5, the last term 89, the common difference 7, required the number of terms.

$$n=\frac{L-a}{d}+1=\frac{89-5}{7}+1=13. \quad \textit{Ans.}$$

Having, therefore, any three of the four terms given, the other may be found, as we have demonstrated above, by the following

FORMULÆ.

(1.) To find the last term.

$$L=a+(n-1)d.$$

(2.) To find the first term.

$$a=L-(n-1)d.$$

(3.) To find the common difference.

$$d=\frac{L-a}{n-1}.$$

(4.) To find the number of terms.

$$n=\frac{L-a}{d}+1.$$

When the series are descending, the unknown difference is a minus quantity in the 1st and 2d formulæ; thus, $-d$.

13. A man travelled 10 days; the first day he went 8 miles, the second day 13 miles, and thus increased his distance each day 5 miles. How far did he travel the last day?

Ans. 53 miles.

14. John Smith's family expenses for the first year were $500; but, after he had been married 12 years, he found his last year's expenses to have been $1325. By how much did he increase his expenses yearly? *Ans.* $75.

15. A man set out from Boston to travel into the country; the first day he travelled 12 miles, the second day 9 miles, the third day 6 miles, and thus continued to travel each day 3 miles less than the preceding. How far did he go the tenth day? *Ans.* —15 miles.

264. To find the sum of the series.

ARITHMETICAL SERIES.

	(1)	(2)	(3)	(4)	(5)	(6)	
Let	2,	5,	8,	11,	14,	17,	be the series.
And	17,	14,	11,	8,	5,	2,	same series inverted.
	19,	19,	19,	19,	19,	19,	sum of both series.

LITERAL SERIES.

	(1)	(2)	(3)	(4)	(5)	(6)	
Let	a,	$a+d$,	$a+2d$,	$a+3d$,	$a+4d$,	$a+5d$	be a series.
And	$a+5d$,	$a+4d$,	$a+3d$,	$a+2d$,	$a+d$,	a	same series inverted.

$2a+5d$, $2a+5d$, $2a+5d$, $2a+5d$, $2a+5d$, $2a+5d$, sum of both series.

We perceive, from the above arithmetical and literal series, that the sum of the extremes is equal to the sum of any two of the means equally distant from each extreme; and that, by adding the two series in their present arrangement, we have the same number for the same successive terms; also, that the sum of both series is twice the sum of either series. Therefore, if 19, the sum of the extremes in the arithmetical series, be multiplied by 6, the number of terms, the product will be the sum of both series. Thus, $19\times6=114$, sum of both series. Therefore, $114\div2=57$ will be the sum of either series.

Again, $2a+5d$ is the sum of the extremes in the *literal* series; and, if this sum be multiplied by 6, the number of terms, the product will be the sum of both series. Thus, $(2a+5d)6=12a+30d$, sum of both series. And $(12a+30d)\div2=6a+15d$, the sum of either series.

Therefore, in all cases, we find that the sum of the series is equal to the sum of the extremes multiplied by half the number of terms; or, the number of terms multiplied by half the sum of the extremes.

If, therefore, the sum of any series be denoted by S, the first term by a, the last term by L, and the number of terms by n, the following will be the formula for obtaining its value:

$$S=\left(\frac{L+a}{2}\right)n.$$

Therefore, if the extremes and the number of terms are given to find the sum of the series, we adopt the following

RULE. *Multiply half the sum of the extremes by the number of terms.*

The two following formulæ, or equations, contain five quantities: a, the first term of a progression; L, the last term; d, the common difference; n, the number of terms; and S, the sum of the series.

If any three of these be given, the other two may be obtained.

(1.) $L=a+(n-1)d.$ (2.) $S=\left(\frac{L+a}{2}\right)n.$

265. The pupil will find that twenty different cases may arise which may be solved by different combinations of the above equations.

To find n in the last equation.

$$\left(S=\frac{L+a}{2}\right)n.$$

By multiplication, $2S=(L+a)n.$

By division, $\frac{2S}{L+a}=n.$

Therefore, $n=\frac{2S}{L+a}.$

If, therefore, the extremes and the sum of the series are given to find the number of terms, we divide twice the sum of the series by the sum of the extremes.

16. Let the extremes be 3 and 39, and the sum of the series 399, to find the number of terms.

$$n=\frac{2S}{L+a}=\frac{2\times399}{39+3}=19. \quad \textit{Ans.}$$

266. To find the last term, L, from the second equation.

$$S=\left(\frac{L+a}{2}\right)n.$$

By multiplication, $2S=(L+a)n.$

By division, $\frac{2S}{n}=L+a.$

By transposition, $\frac{2S}{n}-a=L.$

By transposition of terms, $L=\frac{2S}{n}-a.$

Therefore, having the first term, number of terms, and sum of the series, given to find the last term, we divide twice the sum of the series by the number of terms, and subtract the first term from the quotient.

267. To find the first term, a, from the second equation.

$$S=\left(\frac{L+a}{2}\right)n.$$

Multiplying, $2S=(L+a)n.$

Dividing, $\frac{2S}{n}=L+a.$

Transposing, $\frac{2S}{n}-L=a.$

Changing terms, $a=\frac{2S}{n}-L.$

Therefore, having the last term, number of terms, and sum of the series, given to find the first term, we divide twice the sum of the series by the number of terms, and subtract the last term from the quotient.

17. Let the last term be 39, number of terms 19, and the sum of the series 399, to find the first term.

$$a=\frac{2S}{n}-L=\frac{2\times 399}{19}-39=3. \quad \textit{Ans.}$$

268. To find the common difference, d, from the 1st and 2d equation.

We find the value of L, in the first equation, to be

$$L=a+(n-1)d.$$

Substituting this value of L for S in the 2d equation, and then transposing, we have

$$d=\frac{2S-2an}{n(n-1)}.$$

18. If the first term is 5, the number of terms 15, and the sum of the series 285, what is the common difference?

Ans. 2.

19. If the first term is 3, the number of terms 19, and the sum of the series 399, what is the common difference?

Ans. 2.

20. If the first term is 7, the number of terms 8, and the sum of the series 100, what is the common difference?

Ans. $1\frac{4}{7}$.

PROBLEMS.

1. The first term is 5, the common difference 3. What is the 7th term? *Ans.* 23.

2. The first term is 3, the common difference $4\frac{1}{3}$. What is the 5th term? *Ans.* $20\frac{1}{3}$.

3. The first term is 18, the common difference $\frac{1}{4}$. What is the 7th term? *Ans.* $19\frac{1}{2}$.

4. The first term is 7, the common difference $2\frac{1}{2}$, and the number of terms 5. Required the last term. *Ans.* 17.

5. The first term is $\frac{3}{4}$, the common difference $\frac{4}{5}$. What is the 10th term? *Ans.* $7\frac{19}{20}$.

6. The first term is 0, the common difference $1\frac{1}{2}$. What is the 20th term? *Ans.* $28\frac{1}{2}$.

7. The first term is 10, the common difference -2. What is the 4th term? *Ans.* 4.

8. The first term is -8, the common difference -3. What is the 10th term? *Ans.* -35.

9. The first term of a descending series is 85, common difference 7. Required the 10th term. *Ans.* 22.

10. The first term is $3\frac{1}{3}$, the common difference $2\frac{1}{4}$. What is the 5th term, and the sum of the series? *Ans.* $12\frac{1}{3}$, and $39\frac{1}{6}$.

11. The first term in a descending series is $2\frac{1}{2}$, the common difference is $\frac{1}{4}$. What is the 10th term, and the sum of the series? *Ans.* $\frac{1}{4}$, and $13\frac{3}{4}$.

12. The first term is a, the common difference is d. What is the nth term? *Ans.* $a+d(n-1)$.

13. What is the sum of the odd numbers from 1 to 100? *Ans.* 2500.

14. If the first term is $4\frac{1}{2}$, the common difference $3\frac{1}{2}$, and number of terms 8, what is the sum of the series? *Ans.* 134.

15. If the first term is 7, the common difference -4, and the number of terms 6, what is the sum of the series? *Ans.* -18.

16. If the first term is 5, the last term 19, and the number of terms 6, what are the other terms of the progression? *Ans.* $7\frac{4}{5}$, $10\frac{3}{5}$, $13\frac{2}{5}$, $16\frac{1}{5}$.

17. If the extremes are -9 and 18, and the number of terms 5, what are the other terms of the progression?

Ans. $-2\frac{1}{4}$, $4\frac{1}{2}$, $11\frac{1}{4}$.

18. If the last term of an ascending series is 20, the common difference 5, and the number of terms 8, what is the sum of the series? *Ans.* 20.

19. There is a number consisting of three digits in arithmetical progression, whose sum is 12; and, if 396 be added to the number, the digits will be inverted. What is the number?

Ans. 246.

20. There is a certain island 50 miles in circumference. Two men, A and B, set out to travel round it. A goes 10 miles each day. B goes 2 miles the first day, 5 miles the second day, and 8 miles the third day, travelling each day 3 miles further than the day preceding. How far will A and B be apart the 8th day? *Ans.* 30 miles.

21. John Smith and John Jones set out from Boston for the city of Washington, the distance being 440 miles. Smith started 5 days before Jones, and travels 15 miles per day. Jones travels 25 miles the first day, 23 miles the second day, and 21 miles the third day, travelling each day 2 miles less than the preceding. How far apart will Smith be from Jones at the end of the 20th day, and how far will each be from Washington?

Ans. 135 miles apart. Smith 140 miles from Washington. Jones 275 miles from Washington.

22. If the first term is $\frac{1}{2}$, the common difference $-\frac{1}{6}$, and the number of terms 20, what are the last term and the sum of the series?

$$Ans. \begin{cases} \text{Last term, } -2\frac{2}{3}. \\ \text{Sum of the series, } -21\frac{2}{3}. \end{cases}$$

23. If one extreme is $\frac{1}{3}$, the common difference $-\frac{1}{12}$, and the sum of the series $-1\frac{1}{2}$, what is the number of terms?

Ans. 12.

24. If the first term is $\frac{7}{12}$, last term $2\frac{1}{2}$, and the sum of the series 37, what is the number of terms? *Ans.* 24.

25. If the first term is 3, the last term 17, and the number of terms 29, what are the terms of the series?

Ans. 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, $5\frac{1}{2}$, &c.

26. The sum of the series is $16\frac{1}{4}$, the number of terms 10, and the common difference $\frac{1}{4}$, to find the first term. *Ans.* $\frac{1}{2}$.

27. The first term of an arithmetical series is —5, the common difference $1\frac{1}{2}$; what is the 9th term? *Ans.* 7.

28. What are the three means between —1 and 15?

Ans. 3, 7, and 11.

29. The first term is $1\frac{1}{4}$, number of terms 10, and the sum of the series $6\frac{7}{8}$. What is the common difference? *Ans.* $-\frac{1}{8}$.

30. There are three numbers in arithmetical progression whose sum is 10, and the product of the second and third is $33\frac{1}{3}$. What are those numbers? *Ans.* $-3\frac{1}{3}$, $3\frac{1}{3}$, and 10.

31. The number of terms of an arithmetical progression is equal to $\frac{1}{2}$ the common difference, the last term is equal to 4 times the first, and the sum of the series is equal to $\frac{3}{4}$ the square of the first term. What are the series, and the sum of the series?

Ans. { The series, 20, 32, 44, 56, 68, 80.
Sum of the series, 300.

32. There are four numbers in arithmetical progression whose sum is 28, and the sum of whose squares is 216. What are those numbers? *Ans.* 4, 6, 8, and 10.

33. Find three numbers in arithmetical progression whose sum is 9, and the sum of whose cubes is 99.

Ans. 2, 3, and 4.

34. What are those four numbers in arithmetical progression the sum of the squares of whose first two terms is 34, and the sum of the squares of the last two is 130?

Ans. 3, 5, 7, and 9.

35. A certain number consists of three digits, which are in arithmetical progression; and, if the number be divided by the sum of its digits, the quotient will be $27\frac{4}{7}$, but, if 396 be added

to the number, the digits will be inverted. Required the number. *Ans.* 579.

36. What are those four numbers in arithmetical progression the sum of the squares of whose extremes is 90, and the sum of the squares of the means is 74? *Ans.* 3, 5, 7, and 9.

37. What are those four numbers in arithmetical progression whose sum is 14, and whose continued product is 120?
Ans. 2, 3, 4, and 5.

38. There are four numbers in arithmetical progression, the product of whose extremes is 112, and that of the means 120. What are the numbers? *Ans.* 8, 10, 12, and 14.

39. A and B, 165 miles from each other, set out with a design to meet. A travels one mile the first day, two the second, three the third, and so on. B travels 20 miles the first day, 18 the second, 16 the third, and so on. How soon will they meet? *Ans.* 10 days, or 33 days.

40. There are four numbers in arithmetical progression, whose continued product is 1680, and common difference is 4. Required the numbers. *Ans.* 14, 10, 6, 2.

41. Five persons undertake to reap a field of 87 acres. The five terms of an arithmetical progression, whose sum is 20, will express the times in which they can severally reap an acre, and they all together can finish the job in 60 days. In how many days can each, separately, reap an acre?
Ans. 2, 3, 4, 5, 6 days.

42. A gentleman set out from Boston for New York. He travelled 25 miles the first day, 20 miles the second day, each day travelling 5 miles less than the preceding. How far was he from Boston at the end of the eleventh day? *Ans.*

43. Suppose a number of stones were laid a rod distant from each other for twenty miles, and the first stone a rod from a basket. What length of ground will that man travel over, who gathers them up singly, returning with them, one by one, to the basket? *Ans.* 128,060 miles, 2 rods.

There are twenty different cases in Arithmetical Progression, all of which are exhibited in the following TABLE.

No.	Given.	Requir'd.	Formulæ.
1	a, d, n	l	$l=a+(n-1)d.$
2	a, d, S		$l=-\frac{1}{2}d\pm\sqrt{2dS+(a-\frac{1}{2}d)^2}.$
3	a, n, S		$l=\frac{2S}{n}-a.$
4	d, n, S		$l=\frac{S}{n}+\frac{(n-1)d}{2}.$
5	a, d, n	S	$S=\frac{1}{2}n[2a+(n-1)d].$
6	a, d, l		$S=\frac{l+a}{2}+\frac{l^2-a^2}{2d}.$
7	a, l, n		$S=\left(\frac{l+a}{2}\right)n.$
8	d, n, l		$S=\frac{1}{2}n(2l-(n-1)d).$
9	a, n, l	d	$d=\frac{l-a}{n-1}.$
10	a, n, S		$d=\frac{2S-2an}{n(n-1)}.$
11	a, l, S		$d=\frac{l^2-a^2}{2S-l-a}.$
12	n, l, S		$d=\frac{2nl-2S}{n(n-1)}.$
13	d, n, l	a	$a=l-(n-1)d.$
14	d, n, S		$a=\frac{S}{n}-\frac{(n-1)d}{2}.$
15	d, l, S		$a=\frac{1}{2}d\pm\sqrt{(l+\frac{1}{2}d)^2-2dS}.$
16	n, l, S		$a=\frac{2S}{n}-l.$
17	a, d, l	n	$n=\frac{l-a}{d}+1.$
18	a, d, S		$n=\pm\frac{\sqrt{(2a-d)^2+8dS}-2a+d}{2d}.$
19	a, l, S		$n=\frac{2S}{l+a}.$
20	d, l, S		$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d}.$

SECTION XXI.

GEOMETRICAL PROGRESSION, OR PROGRESSION BY QUOTIENT.

Art. 269. When there are three or more numbers, such that the same quotient is obtained by dividing the second by the first, and the third by the second, and the fourth by the third, &c.; or, such that they increase or decrease by a constant multiplier, they are said to be in Geometrical Progression, and are called a Geometrical Series. Thus,

	(1)	(2)	(3)	(4)	(5)	(6)	
(1.)	2,	6,	18,	54,	162,	486	= 728, sum of the series.
(2.)	486,	162,	54,	18,	6,	2	= 728, sum of the series.

The first is called an ascending series, and the second a descending series.

In the first the quotient or multiplier is 3, and it is called the *ratio*. In the second the ratio is $\frac{1}{3}$.

270. The first and last terms of a series are called the *extremes*, and the others are the means.

271. It will readily be perceived, in either of the above series that the product of the extremes is equal to the product of any two of the means equally distant from the extremes. Thus, $2\times486=6\times162=18\times54=972$.

272. If there are only three terms, the product of the extremes is equal to the square of the second term.

273. It is evident, by examining either the above series, that any term may be obtained by multiplying the first term by the ratio as many times, wanting one, as there are terms required.

If, therefore, the 1st term is 2, and the ratio 3, and we wish to obtain the 6th term, we have only to multiply the 1st term, 2, by the ratio 3, five times.

Thus, $2\times3\times3\times3\times3\times3=486$, the 6th term.

The above may be generalized in the following manner:

Let a = first term of a series.
L = the last term.
r = the ratio.
n = the number of terms.
S = the sum of the series.

(1) (2) (3) (4) (5) (6)
Then a, ar, ar^2, ar^3, ar^4, ar^5, &c., may represent any geometrical series; and, if r, the ratio, is considered as more than a unit, the series is *ascending;* but, if r is less than a unit, the series is *descending*.

The exponent of r in the second term is 1, in the third term 2, in the fourth term 3, in the fifth term 4, and so on; therefore, the exponent of r in the last term will always be *one* less than the number of terms. The exponent of the nth term in the above series would therefore be ar^{n-1}.

274. If, therefore, in any series the number of terms be denoted by n, and the last term by L, the following will be the formula for finding the last term:

(1.) $L=ar^{n-1}$.

And $L=r^{n-1}$, when the first term is a unit.

In the above equation we have four quantities, a, L, r, and n; and, if any three of them be given, the others may be obtained as follows:

To find a, the first term, we divide both terms of the above equation by r^{n-1}, and transpose the terms; and we have

(2.) $$a=\frac{L}{r^{n-1}}.$$

To obtain r, the ratio, we divide the terms of the 1st equation by a, extract the $(n-1)$th root, and transpose the terms; and we have

(3.) $$r=\sqrt[n-1]{\frac{L}{a}}.$$

To find n, we shall show when we come to treat of exponential quantities.

EXAMPLES.

1. If the first term is 7, the ratio 3, and the number of terms 5, required the last term.

$$L=ar^{n-1}=7(3)^4=567.\quad \textit{Ans.}$$

2. If the first term is 1, the ratio 5, and the number of terms 5, what is the last term?

$$L=r^{n-1}=5^4=625.\quad \textit{Ans.}$$

3. If the last term is 405, the ratio 3, and the number of terms 5, what is the first term?

$$a=\frac{L}{r^{n-1}}=\frac{405}{3^{5-1}}=5.\quad \textit{Ans.}$$

4. If the last term is 8, ratio 5, and the number of terms 4, what is the first term?

$$a=\frac{L}{r^{n-1}}=\frac{8}{5^{4-1}}=\frac{8}{125}.\quad \textit{Ans.}$$

5. If the first term is 5, the last term 1215, and the number of terms 6, what is the ratio?

$$r=\left(\frac{L}{a}\right)^{\frac{1}{n-1}}=\left(\frac{1215}{5}\right)^{\frac{1}{6-1}}=243^{\frac{1}{5}}=3.\quad \textit{Ans.}$$

6. If the first term is $\frac{1}{5}$, the last term $\frac{27}{320}$, and the number of terms 4, what is the ratio?

$$r=\left(\frac{L}{a}\right)^{\frac{1}{n-1}}=\left(\frac{\frac{27}{320}}{\frac{1}{5}}\right)^{\frac{1}{4-1}}=\left(\tfrac{27}{320}\times\tfrac{5}{1}\right)^{\frac{1}{4-1}}=\left(\tfrac{27}{64}\right)^{\frac{1}{3}}=\tfrac{3}{4}.\quad \textit{Ans.}$$

7. If the first term is $\frac{1}{16}$, the last term 64, and the number of terms 6, required the ratio. *Ans.* 4.

8. If the last term is 135, the number of terms 4, the ratio 3, what is the first term? *Ans.* 5.

275. To find any number of geometrical means between any two given numbers.

In the 3d formula, we found $r=\sqrt[n-1]{\frac{L}{a}}$.

If we let m represent the number of means, then $m+2=n$, for the number of terms is always two more than the number of means.

Therefore, $$\left(\frac{L}{a}\right)^{\frac{1}{n-1}}=\left(\frac{L}{a}\right)^{\frac{1}{m+1}}.$$

Consequently, $$r=\left(\frac{L}{a}\right)^{\frac{1}{m+1}}.$$

276. Having, therefore, the extremes given to find any number of means, we divide the greater extreme or number by the less extreme, and extract that root of the quotient denoted by the number of means plus 1. This root is the ratio; and having the ratio, the means are readily obtained.

EXAMPLES.

9. Find two geometrical means between 6 and 162.

$162\div6=27$: $\sqrt[3]{27}=3$, the ratio; $6\times3=18$, the first mean; $18\times3=54$, the second mean.

10. What is the geometrical mean between 18 and 882?

$882\div18=49$: $\sqrt{49}=7$, the ratio; $18\times7=126$, the geometrical mean.

11. Required the five geometrical means between 1 and 64.
Ans. 2, 4, 8, 16, 32.

12. A has a piece of land, which is 18 rods wide, and 288 rods long. Required the side of a square piece that shall contain an equal number of square rods. *Ans.* 72 rods.

277. To find the sum of all the terms of a geometrical series.

Let the following be the series:

(1.) 2, 6, 18, 54, 162.

By examining this series, we find the first term 2, the ratio 3, and the last term 162.

If we multiply each term in the series by the ratio 3, we obtain

(2.) 6, 18, 54, 162, 486.

It is evident that the sum of this last series is three times the former; therefore the difference between them will be equal to twice the sum of the first series. Thus,

From 6, 18, 54, 162, 486, second series,
Take 2, 6, 18, 54, 162, first series.

−2 486=484, difference of the series.

From the above operation, it appears that 484 is twice the sum of the first series; and, therefore, $484 \div 2 = 242$ is the sum required.

By examining the process, we perceive that 242 is obtained by multiplying the last term of the first series, 162, by the ratio 3, and subtracting from the product the first term 2, and dividing the remainder, 484, by 2 a number which is *one* less than the ratio. Hence the propriety of the following

Rule. *Multiply the last term by the ratio, find the difference between this product and the first term, divide this remainder by the difference between the ratio and unity, and we have the sum of the series.*

278. We may generalize the above, as follows:

Let a represent the first term of a geometrical series, r the ratio, L the last term, n the number of terms, and S the sum of the series. Then

$$(1.)\qquad S = a + ar + ar^2 + ar^3 + ar^4 + ar^5.$$

We next multiply each term of the above equation by r, and we have

$$(2.)\qquad Sr = ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6.$$

By subtracting the first equation from the second, we have

$$Sr - S = ar^6 - a.$$

Dividing by $r-1$, we have the formula for finding the sum of the series

$$S = \frac{ar^6 - a}{r-1}, \text{ or } \frac{ar^n - a}{r-1}, \text{ or } a\,\frac{(r^n - 1)}{r-1}.$$

If the ratio is less than a unit, we transpose the terms, thus:

$$S = \frac{a - ar^6}{1-r}, \text{ or } \frac{a - ar^n}{1-r}, = a\frac{(1-r^n)}{1-r}.$$

279. The index of the ratio is always equal to the number of terms.

By the above formulæ, we have a method for finding the sum of the series without the last term, which may be expressed by the following

RULE. *Raise the ratio to a power whose exponent is equal to the number of terms; multiply this power by the first term, find the difference between this product and the first term, and divide this remainder by the difference between the ratio and unity.*

If we substitute the value of L as found in Art. 274, we shall have

$$S=\frac{Lr-a}{r-1}.$$

A rule for this formula would be the same as in Art. 278.

13. If the first term is 7, the ratio 3, and the number of terms 5, what is the sum of the series?

$$S=\frac{ar^n-a}{r-1}=\frac{7\times3^5-7}{3-1}=847. \quad \textit{Ans.}$$

14. If the first term is 9, the ratio $\frac{2}{5}$, and the number of terms 4, what is the sum of the series?

$$S=\frac{a-ar^n}{1-r}=\frac{9-(9\times(\frac{2}{5})^4}{1-\frac{2}{5}}=14\tfrac{77}{125}. \quad \textit{Ans.}$$

15. If the first term is 144, the ratio 1.06, and the number of terms 4, what is the sum of the series? *Ans.* 629.945.

16. If the first term is 9, the ratio $\frac{1}{4}$, the number of terms 6, what is the sum of the series? *Ans.* $11\frac{1021}{1024}$.

17. If the first term is a, the ratio r, and the number of terms n, required the sum of the series.

$$\textit{Ans.} \quad \frac{ar^n-a}{r-1}=\frac{a(r^n-1)}{r-1}.$$

18. If the first term is 1, the ratio 2, and the number of terms 7, what is the sum of the series? *Ans.* 127.

19. If the first term is 5, the ratio 10, and the number of terms 7, what is the sum of the series? *Ans.* 5555555.

20. If the first term is 4, the ratio $\frac{1}{4}$, and the number of terms 5, what is the sum of the series? *Ans.* $5\frac{21}{64}$.

21. If the first term is 5, the ratio $\frac{1}{5}$, and the number of terms 5, what is the sum of the series? *Ans.* $6\frac{31}{125}$.

22. A gentleman agreed with another to board him for 9 days; he was to pay 3 cents for the first day's board, 9 cents for the second day, 27 cents for the third day, and so on, in this ratio. What was the amount of the bill for the gentleman's board? *Ans.* \$295.23.

To find L, r, and a, from the following equation.

$$S=\frac{Lr-a}{r-1}.$$

Multiplying by $r-1$, $Sr-S=Lr-a.$

Resolving into factors, $S(r-1)=Lr-a.$

Transposition, $Lr=S(r-1)+a.$

Division, $L=\frac{S(r-1)+a}{r}.$

To find r from the above equation.

$$S=\frac{Lr-a}{r-1}.$$

Multiplying by $r-1$, $Sr-S=Lr-a.$

Transposing, $Sr-Lr=S-a.$

Dividing by $S-L$, $r=\frac{S-a}{S-L}.$

To find a from the above equation.

$$S=\frac{Lr-a}{r-1}.$$

Multiplying by $r-1$, $Sr-S=Lr-a.$

Transposing, $a=Lr-(r-1)S.$

23. If the first term is 3, the ratio 2, and the sum of the series 93, what is the last term? *Ans.* 48.

24. Insert three geometrical means between $\frac{1}{2}$ and 128.
Ans. 2, 8, 32.

25. If the first term is 2, the last term 4374, and the number of terms 8, what is the ratio? *Ans.* 3.

26. If the ratio is 2, the number of terms 6, and the greatest term 128, what is the least term? *Ans.* 4.

27. If the first term is $3\frac{1}{3}$, the ratio $\frac{3}{5}$, the number of terms 8, what is the last term, and what is the sum of the series?
Ans. Last term $\frac{1458}{15625}$, and the sum of series $8\frac{9064}{46875}$.

28. If the first term is 1, the last term 64, and the number of terms 7, what are the ratio, and the sum of the series?
Ans. Ratio, 2; the sum of the series, 127.

29. If the last term is 64, the number of terms 7, and the sum of the series 127, what are the ratio, and the first term?
Ans. Ratio, 2; the first term, 1.

30. If the first term is 2, the ratio 4, and the number of terms 12, what are the last term, and the sum of the series?
Ans. Last term, 8388608; sum of the series, 11184810.

31. The product of three terms in geometrical progression is 64, and the sum of their cubes is 584. What are those numbers? *Ans.* 2, 4, 8.

32. There are four numbers in geometrical progression, the second of which is less than the fourth by 24, and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers. *Ans.* 1, 3, 9, 27.

33. It is required to find four numbers in geometrical progression, such that the difference of the two means shall be 14, and the difference of the extremes 49.
Ans. 7, 14, 28, and 56.

The following are the two fundamental equations from which the twenty different cases are exhibited, —

$$L = ar^{n-1},$$

$$S = \frac{Lr - a}{r - 1},$$

and which are found in the following

TABLE.

No	Given.	Requir'd.	Formulæ.
1	a, r, n	l	$l=ar^{n-1}.$
2	a, r, S		$l=\frac{a+(r-1)S}{r}.$
3	a, n, S		$l(S-l)^{n-1}=a(S-a)^{n-1}.$
4	r, n, S		$l=\frac{(r-1)Sr^{n-1}}{r^n-1}.$
5	a, r, n	S	$S=\frac{ar^n-a}{r-1}.$
6	a, r, l		$S=\frac{lr-a}{r-1}.$
7	a, n, l		$S=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}.$
8	r, n, l		$S=\frac{lr^n-l}{r^n-r^{n-1}}.$
9	a, n, l	r	$r=\sqrt[n-1]{\frac{l}{a}}.$
10	a, n, S		$ar^n-rS=a-S.$
11	a, l, S		$r=\frac{S-a}{S-l}.$
12	n, l, S		$(S-l)r^n-Sr^{n-1}=-l.$
13	r, n, l	a	$a=\frac{l}{r^{n-1}}.$
14	r, n, S		$a=\frac{(r-1)S}{r^n-1}.$
15	r, l, S		$a=lr-(r-1)S.$
16	n, l, S		$a(S-a)^{n-1}=l(S-l)^{n-1}.$
17	a, r, l	n	$n=\frac{\log. l-\log. a}{\log. r}+1.$
18	a, r, S		$n=\frac{\log. [a+(r-1)S]-\log. a}{\log. r}.$
19	a, l, S		$n=\frac{\log. l-\log. a}{\log. (S-a)-\log. (S-l)}+1.$
20	r, l, S		$n=\frac{\log. l-\log. [lr-(r-1)S]}{\log. r}+1.$

The last four cases in the preceding table can be performed only by the aid of logarithms, as they belong to exponential or transcendental equations. They will, therefore, receive attention in their proper place.

SECTION XXII.

HARMONICAL PROGRESSION.

Art. 280. Three numbers are said to be in harmonical progression when the first is to the third as the difference between the first and second is to the difference between the second and third.

Thus the numbers 3, 4, 6, are in harmonical proportion.

For $3 : 6 :: 4-3 : 6-4$.

Or a, b, c, are in harmonical proportion when

$$a : c :: b-a : c-b.$$

Thus, if the length of three strings of a musical instrument be as the numbers 3, 4, 6, they will sound an octave 3 to 6, a fifth 2 to 3, and a fourth 3 to 4.

281. Four numbers are in harmonical proportion when the first is to the fourth as the difference between the first and second is to the difference between the third and fourth. Thus the numbers 5, 6, 8, 10, are in harmonic proportion.

For $5 : 10 :: 6-5 : 10-8$.

Strings of such lengths will sound an octave 5 to 10, a sixth greater 6 to 10, a third greater 8 to 10, a third less 5 to 8, and a fourth 6 to 8.

282. Any number of quantities, a, b, c, d, e, &c., are in harmonical progression if $a : c :: a-b : b-c$; $b : d :: b-c : c-d$; $c : e :: c-d : d-e$, &c.

283. The reciprocal quantities in harmonical progression are in arithmetical progression.

Thus, if a, b, c, d, e, &c., are in harmonical progression, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}$, &c., will be in arithmetical progression.

SECTION XXIII.

INFINITE SERIES.

Art. **284.** An infinite decreasing geometrical series is one whose ratio is less than unity, and the number of whose terms is infinite.

To find the sum of an infinite series decreasing in geometrical progression.

We have already found, Art. 277, that the sum of a descending series in geometrical progression may be ascertained by the following formula.

$$S=\frac{a-ar^n}{1-r}, \text{ or } S=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

285. Now, if r^n be a fraction less than a unit, it is evident that the greater the number n, the smaller will be the quantity r^n. If, therefore, a great number of terms of a descending series be taken, the quantity r^n will be very small; and, if we suppose n greater than any assignable number, then the quantity, or its value, may be considered as nothing $=0$.

Hence the latter part of the formula, $-\frac{ar^n}{1-r}$, should be omitted, and it will stand

Thus, $$S=\frac{a}{1-r}.$$

The rule, therefore, for finding the sum of the series, is as follows:

Rule. *Divide the first term by the difference between unity and the ratio.*

EXAMPLES.

1. What is the sum of the infinite series, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, &c.?

$$\frac{1}{1-\frac{1}{3}}=\frac{1}{\frac{2}{3}}=1\tfrac{1}{2}.\quad \textit{Ans.}$$

2. What is the sum of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c., to infinity? *Ans.* 2.

3. What is the sum of the series, 8, $\frac{8}{5}$, $\frac{8}{25}$, $\frac{8}{125}$, &c., carried to infinity? *Ans.* 10.

4. Find the value of $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{12}$, &c., to infinity. *Ans.* $1\frac{1}{3}$.

5. Find the value of 4, 1, $\frac{1}{4}$, $\frac{1}{16}$, &c., to infinity. *Ans.* $5\frac{1}{3}$.

6. What is the exact sum of 1, $\frac{1}{10}$, $\frac{1}{100}$, &c., to infinity? *Ans.* $1\frac{1}{9}$.

7. Find the exact value of the circulating decimal .444, &c., to infinity.

$$.444,\ \&c.=\frac{4}{10}+\frac{4}{100}+\frac{4}{1000},\ \text{the ratio being } \frac{1}{10}.$$

$$\frac{\frac{4}{10}}{1-\frac{1}{10}}=\frac{\frac{4}{10}}{\frac{9}{10}}=\frac{4}{10}\times\frac{10}{9}=\frac{40}{90}=\frac{4}{9}.\quad \textit{Ans.}$$

[See NATIONAL ARITHMETIC, page 128.]

8. What common fraction will exactly express the value of the repeating decimal .454545, &c.?

$$.454545=\frac{45}{100}+\frac{45}{10000}+\frac{45}{1000000},\ \text{the ratio being } \frac{1}{100}.$$

$$\frac{\frac{45}{100}}{1-\frac{1}{100}}=\frac{\frac{45}{100}}{\frac{99}{100}}=\frac{45}{100}\times\frac{100}{99}=\frac{4500}{9900}=\frac{5}{11}.\quad \textit{Ans.}$$

9. What common fraction is the exact value of the decimal $.\dot{5}7142\dot{8}$? *Ans.* $\frac{4}{7}$.

10. What common fraction is the exact value of $.\dot{8}5714\dot{2}$? *Ans.* $\frac{6}{7}$.

11. What is the exact value of $.5\dot{3}$?

$$.5\dot{3}=\frac{5}{10}\ \text{and}\ \frac{3}{100}+\frac{3}{1000}+\frac{3}{10000},\ \&c.$$

$$\frac{\frac{3}{100}}{1-\frac{1}{10}}=\frac{\frac{3}{100}}{\frac{9}{10}}=\frac{3}{100}\times\frac{10}{9}=\frac{30}{900}=\frac{1}{30};\ \frac{5}{10}+\frac{1}{30}=\frac{8}{15}.\quad \textit{Ans.}$$

12. What is the value of $.13\dot{8}$? *Ans.* $\frac{5}{36}$.

13. Find the ratio of an infinite series whose first term is 8, and the sum of the series 10. *Ans.* $\frac{1}{5}$.

14. Find the ratio of an infinite series whose first term is $\frac{2}{3}$, and whose sum is $1\frac{1}{3}$. *Ans.* $\frac{1}{2}$.

15. Find the first term of an infinite progression of which the ratio is $\frac{1}{5}$, and the sum 10. *Ans.* 8.

SECTION XXIV.

SIMPLE INTEREST.

Art. 286. *Interest* is the compensation which the borrower makes to the lender for the use of a certain sum of money for a given time.

Principal is the sum lent.

Rate per cent. is the sum agreed on for the loan of $1, or $100, for one year.

Amount is the sum of the interest and principal.

Legal interest is the rate per cent. established by law.

Let $p =$ principal.
$r =$ rate per cent., written in hundredths.
$t =$ time in years.
$a =$ amount.
i or $a-p =$ interest for the given time.

Hence, if r be the interest of one dollar for one year, it is evident that the interest of p dollars will be p times $r = pr$.

And if pr be the interest of p dollars for one year, it is certain that for t years it will be t times as much, $= ptr$, and that $p+ptr$ will be the amount, and i or $a-p$ will be the interest.

287. Hence, having the principal, rate per cent., and time given, to find the interest and amount, we have the following formulæ:

Formula for the interest,

$$i = ptr.$$

Formula for the amount,

$$a = p + ptr.$$

From the preceding formulæ we have, for finding the interest and amount, the following

RULE. *Multiply the principal by the rate per cent., considered as a decimal, and this product by the time in years, and the result is the interest.*

If there are months and days, let the months be considered as fractions of a year, and the days as fractions of a month.

By adding the interest to the principal, we have the amount.

[See NATIONAL ARITHMETIC, page 164.]

EXAMPLES.

1. What is the interest of $740 for 4 years, at 6 per cent.?

$i=ptr=740\times.06\times4=\$177.60.$ *Ans.*

2. What is the interest of $380 for 10 years, at 5 per cent.? *Ans.* $190.

3. What is the interest of $890.75 for 3 years, 6 months, at 8 per cent.? *Ans.* $249.41.

4. What is the interest of $17.18 for 5 years, 2 months, 10 days, at 4½ per cent.? *Ans.* $4.02.

5. What is the amount of $144 for 3 years, at 8 per cent.?

$a=p+prt=144+(144\times.08\times3)=\$178.56.$ *Ans.*

6. What is the amount of $800 for 6 years, 1 month, 12 days, at 6 per cent.? *Ans.* $1093.60.

7. What is the amount of $670.18 for 3 years, 7 months, 20 days, at 9 per cent.? *Ans.* $889.66.

288. Having the amount, time, and rate per cent. given, to find the principal.

By transposing, &c., the last equation, we have

$$p=\frac{a}{1+tr}.$$

From which we have the following

RULE. *Multiply the time by the rate per cent., and add* 1 *to the product; with this sum divide the amount, and the quotient is the principal.*

8. Received \$472 for a certain sum that had been on interest, at 6 per cent., for 3 years. What was the sum lent?

$$p=\frac{a}{1+tr}=\frac{472}{1+(3\times.06)}=\$400.\quad \textit{Ans.}$$

9. What principal will amount to \$570 in 10 years, at 5 per cent.? *Ans.* \$380.

10. What principal will amount to \$1140.16 in 3 years, 6 months, at 8 per cent.? *Ans.* \$890.75.

11. Lent a certain sum for 5 years, 2 months, 10 days, at $4\frac{1}{2}$ per cent., and received interest and principal \$21.20; what was the sum lent? *Ans.* \$17.18.

12. My friend borrowed of me a certain sum, which he kept 3 years, and for which I charged him 8 per cent., and received interest and principal \$178.56. What was the sum I lent him? *Ans.* \$144.

13. Received as interest and principal \$889.66 from a friend to whom I had loaned a certain sum for 3 years, 7 months, and 20 days, at 9 per cent. What was the consideration of his note? *Ans.* \$670.18.

289. Having the amount, principal, and rate per cent. given, to find the time.

By transposing and reducing the last equation, we have the following formula for finding the time, t.

$$t=\frac{a-p}{rp}=\frac{i}{rp}.$$

From the above formula we have the following

RULE. *Divide the interest by the product of the principal multiplied by the rate per cent., and the quotient is the time.*

[See NATIONAL ARITHMETIC, page 181.]

14. How long will it require \$300 to amount to \$372, at 6 per cent.?

Let $$t=\frac{a-p}{rp}=\frac{372-300}{.06\times300}=4 \text{ years.}\quad \textit{Ans.}$$

15. In what time will \$380 amount to \$570, at 5 per cent.? *Ans.* 10 years.

16. Lent, at 8 per cent., $890.75, for which I received $1140.16; for how long time was the money lent?

Ans. 3 years, 6 months.

17. For $17.18, which was loaned at 4½ per cent., there was received $21.20. For how long time had it been lent?

Ans. 5 years, 2 months, 10 days.

18. The interest and principal, on a certain sum, at 9 per cent., are $889.66; and the interest is $670.18 less than the amount. How long was the money at interest?

Ans. 3 years, 7 months, 20 days.

19. A has B's note, dated January 1, 1851, for $320, at 9 per cent. When will the note amount to $353.60?

Ans. March 1, 1852.

290. Having the principal, interest and time given, to find the rate per cent.

By transposing the last formula, we obtain the following for finding r, the rate per cent. Thus,

$$r=\frac{a-p}{pt}, \text{ or } \frac{i}{pt}.$$

The pupil will perceive that the amount is known when the interest and principal are given.

What is the rate per cent. for $300, that it shall amount to $372 in 4 years?

$$r=\frac{a-p}{pt}=\frac{372-300}{300\times4}=.06, \text{ or 6 per cent.}$$

Hence we deduce the following

RULE. *Divide the interest by the product of the principal multiplied by the time, and the quotient is the rate per cent.*

20. If $380 amount to $570 in ten years, what is the rate per cent.? *Ans.* 5 per cent.

21. Lent $890.75, for 3 years, 6 months, and received for the amount $1140.16. What was the rate per cent.?

Ans. 8 per cent.

22. If $17.18 amount to $21.20 in 5 years, 2 months, and 10 days, what is the rate per cent.? *Ans.* 4½ per cent

23. If the interest of \$670.18 for 3 years, 7 months, and 20 days, be \$219.48, what is the rate per cent.?

Ans. 9 per cent.

24. John Smith, Jr., gave me his note, dated January 1, 1848, for \$144: but he having been unfortunate in business, I agreed, May 7, 1851, to give him up his note for \$153.64.8. What per cent. did I receive? *Ans.* 2 per cent.

25. My tailor informs me that my "freedom suit" will require $7\frac{1}{2}$ square yards of cloth; but the cloth I am about to purchase will shrink 5 per cent. in width, and 4 per cent. in length, and the cloth is 60 inches wide. How many yards must I purchase? *Ans.* 4 yards, $33\frac{12}{19}$ inches.

SECTION XXV.

DISCOUNT AT SIMPLE INTEREST.

Art. 291. Discount is an allowance for the payment of any sum of money before it becomes due, and is the difference between that sum and its present worth.

The present worth of any sum due some time hence is such a sum as, if put at interest, would, in the time for which the discount is to be made, amount to the sum then due.

To find the worth of any sum due at any time hence:

Let S = the sum due.
p = the present worth.
t = the time in years.
r = the rate per cent. considered as so many hundredths.

We have before shown, in Art. 287, that $a=p+ptr$.

We now substitute S for a, and consider p to represent the present worth; and, by transposing the equation, find

$$p=\frac{S}{1+tr};$$

from which we deduce the following

RULE. *Multiply the time by the rate per cent., add* 1 *to the product, and divide the sum on which the discount is to be taken by this sum, and the quotient is the present worth.*

If the present worth is taken from the sum due, the remainder is the discount.

[See NATIONAL ARITHMETIC, page 187.]

1. What is the present value of $500, due 4 years hence, at 6 per cent.?

$$p=\frac{S}{1+tr}=\frac{500}{1+(4\times.06)}=\$403.22+. \quad \textit{Ans.}$$

By transposing the quantities in the above formula, we may obtain the values of s, t, and r.

2. What is the present worth of $372, due 4 years hence, at 6 per cent.? *Ans.* $300.

3. What is the present worth of $133.20, due 20 months hence, at 8¼ per cent.? *Ans.* $117.09.

4. What is the discount on $21.20, due 5 years, 2 months, 10 days hence, at 4½ per cent.? *Ans.* $4.02.

5. A has B's note, dated January 1, 1851, for $353.60, to be paid March 1, 1852, without interest. What was the value of this note at the time it was given, if 9 per cent. discount is allowed? *Ans.* $320.

6. Which is worth the most, A's note for $144, due 10 years hence, at 6 per cent., or B's note for $176.40, due 8 years hence, at 12 per cent.? *Ans.*

7. A legacy of $1725 is due one year hence. What is its present value, at 15 per cent.? *Ans.* $1500.

8. James Brown has S. Smith's note for $162, payable 6

months hence; but Brown, being obliged to raise money, sold the note for $150. What per cent. did he allow?

Ans. 16 per cent.

9. Bought a farm for $590, for which I was to pay in a certain time, without interest; but, by making prompt payment, I was allowed a discount of 6 per cent. for the whole time, and paid only $500. How long was the time allowed for payment?

Ans. 3 years.

10. Bought a horse for $200, and gave my note, payable in 60 days. What ready money, at 15 per cent., will discharge the debt? *Ans.* $195.12+.

11. What is the present worth of $1827, due 100 years hence, at 6 per cent.? *Ans.* $261.

SECTION XXVI.

PARTNERSHIP, OR COMPANY BUSINESS.

Art. 292. Partnership is the association of two or more persons in business, with an agreement to share the profits and losses in proportion to the amount of the capital stock contributed by each.

EXAMPLES.

1. Three men, A, B and C, enter into partnership for two years, with a capital of $1600. A puts into the firm $300, B $500, and C $800. They gain $320. What is each man's share of the gain?

Let $x =$ A's gain.

Then, as each man's share of the gain will be in proportion to his stock,

And $\frac{5x}{3} =$ B's gain.

$\frac{8x}{3} =$ C's gain.

And $$x+\frac{5x}{3}+\frac{8x}{3}=\$320.$$

$$3x+5x+8x=960.$$

$$16x=960.$$

$$x=60=\text{A's gain.}$$

$$\frac{5x}{3}=100=\text{B's gain.}$$

$$\frac{8x}{3}=160=\text{C's gain.}$$

VERIFICATION.

$$60+100+160=\$320.$$

Or, let m, n, and p represent A, B, and C's stock, and a the sum gained.

Also, let $x=$ A's gain.

Then, it is evident that each man must receive according to his capital.

That is, as A's stock is to his gain, so will B's stock be to his gain, &c.

Therefore, $$m:x::n:\frac{nx}{m}=\text{B's.}$$

And $$m:x::p:\frac{px}{m}=\text{C's.}$$

Then, $$x+\frac{nx}{m}+\frac{px}{m}=a.$$

And $$mx+nx+px=am.$$

Therefore, $$x=\frac{am}{m+n+p}=\frac{320\times300}{300+500+800}=\$60,\text{ A's gain.}$$

Then, by the principle above stated,

$$m:\frac{am}{m+n+p}::n:\frac{an}{m+n+p}=\frac{320\times500}{300+500+800}=\$100,\text{ B's gain.}$$

And,

$$m:\frac{am}{m+n+p}::p:\frac{ap}{m+n+p}=\frac{320\times800}{300+500+800}=\$160,\text{ C's gain.}$$

VERIFICATION.

$$\frac{am}{m+n+p}+\frac{an}{m+n+p}+\frac{ap}{m+n+p}=\frac{(m+n+p)a}{m+n+p}=a=\$320.$$

Therefore, to find the gain or loss on any man's stock, we deduce from the above formulæ the following

RULE. *Multiply the whole gain by each man's stock, and divide the product by the whole stock.*

293. Having each man's gain, and the amount of stock given, to find each man's share in the stock.

2. A, B, and C, while in trade, gained as follows. A gained \$50, B \$70, and C \$90. The amount of their stock in trade was \$4200. What was the amount of each man's stock?

It is evident that each man's stock was in proportion to his gain.

Let x=A's stock.

Then $\frac{7x}{5}$=B's stock.

And $\frac{9x}{5}$=C's stock.

Therefore, $x+\frac{7x}{5}+\frac{9x}{5}=4200.$

$$5x+7x+9x=21000.$$

$$21x=21000.$$

$x=1000.$ A's stock.

$\frac{7x}{5}=1400.$ B's stock.

$\frac{9x}{5}=1800.$ C's stock.

4200. Proof.

If we change the symbols of the first question, putting m, n, and p, for the gain of each man respectively, and a for the stock, we obtain the following formulæ for finding the amount of each man's stock:

$$\frac{ma}{m+n+p}=\frac{50\times4200}{50+70+90}=\$1000.\quad \text{A's stock.}$$

$$\frac{na}{m+n+p}=\frac{70\times4200}{50+70+90}=\$1400.\quad \text{B's stock.}$$

$$\frac{pa}{m+n+p}=\frac{90\times4200}{50+70+90}=\$1800.\quad \text{C's stock.}$$

Hence, for finding each man's stock, we have the following

RULE. *Multiply the whole stock by each man's gain, and divide the product by the whole gain.*

3. Two men, M and N, engaged in trade. M put in $500, and N $750. They gained $120. What is each man's gain?
Ans. M gained $48, N gained $72.

4. Q and X hired a field for $120, which they used for a pasture. Q put in 11 cows, and X 15 cows. What sum should each man pay? *Ans.* Q pays $\$50.76\frac{12}{13}$, X pays $\$69.23\frac{1}{13}$.

5. A and B purchased a factory for $17,000. A paid $10,000, and B the remainder. They gained $1500. What sum should each receive? *Ans.* A $\$882\frac{6}{17}$, B $\$617\frac{11}{17}$.

6. A, B, and C engaged in trade, with a capital of $6000. They gained $240. A's share of the gain was $100, B's $80, and C's $60. What part of the stock did each own?
Ans. A $2500, B $2000, and C $1500.

7. A, B, and C hire a pasture for the season for $100. A put in 5 horses, B 7 oxen, and C 9 cows. Two horses eat as much as 3 oxen, and 4 oxen eat as much as 5 cows. What part of the expense must each pay? *Ans.* A pays $\$34.56\frac{48}{217}$, B pays $\$32.25\frac{175}{217}$, and C pays $\$33.17\frac{211}{217}$.

8. Three men, A, B, and C, agreed to reap a field that was 40 rods square for $32. A reaped a part that was 25 rods square, B reaped 400 square rods, and C the remainder. What sum did each receive? *Ans.* A $12.50, B $8, C $11.50.

PARTNERSHIP ON TIME, OR DOUBLE FELLOWSHIP.

9. A, B, and C engaged in trade. A put in $2000 for 4 months, B put in $3000 for 8 months, and C put in $4000 for

12 months. They gained \$780. What is each man's share of the gain?

Let m, n, p, represent each man's stock, a the whole gain, and t, t', t'', the time each man's stock was in trade. It is evident that each man's stock gains not only in proportion to its sum, but also in proportion to the time it is in trade. For \$2000 will gain four times as much in four months as it would in one month, and \$2000 for four months is the same as \$8000 for one month. We must, therefore, multiply each man's stock by the time it was in trade. It is therefore evident, that as A's gain is to B's gain, as A's stock multiplied by his time is to B's stock multiplied by his time, &c.

Let x, y, z = A, B, C's gain respectively.

Then $$x : y :: mt : nt'.$$

Multiplying extremes, &c., $$y=\frac{nt'x}{mt}=\text{B's gain.}$$

And $$x : z :: mt : pt''.$$

Multiplying extremes, &c., $$z=\frac{pt''x}{mt}=\text{C's gain.}$$

And $$x+\frac{nt'x}{mt}+\frac{pt''x}{mt}=a.$$

Multiplying by mt, $$mtx+nt'x+pt''x=mta.$$

Therefore,

$$x=\frac{mta}{mt+nt'+pt''}=\frac{2000\times4\times780}{2000\times4+3000\times8+4000\times12}=\$78.\ _{\text{A's gain}}$$

But $$y=\frac{nt'x}{mt}.$$

And by substitution, $$y=\frac{nt'}{mt}\times\frac{mta}{mt+nt'+pt''}=\frac{nt'a}{mt+nt'+pt''}=$$

$$\frac{3000\times8\times780}{2000\times4+3000\times8+4000\times12}=\$234.\ \text{B's gain.}$$

And $$z=\frac{pt''x}{mt}.$$

And by substitution,

$$z=\frac{pt''}{mt}\times\frac{mta}{mt+nt'+pt''}=\frac{pt''a}{mt+nt'+pt''}=$$

$$\frac{4000\times12\times780}{2000\times4+3000\times8+4000\times12}=\$468.\quad \text{C's gain.}$$

The above equations, by dividing the numerators each into two factors, may be expressed by the following proportions:

$$mt+nt'+pt'' : mt :: a : x.$$

$$mt+nt'+pt'' : nt' :: a : y.$$

$$mt+nt'+pt'' : pt'' :: a : z.$$

Hence the following arithmetical

RULE. *Multiply each man's stock by the time it was continued in trade, and then say,* As the sum of all the products is to each man's product, so is the whole gain or loss to each man's gain or loss. [See NATIONAL ARITHMETIC, Sec. LVI.]

10. A commenced business January 1, 1850, with a capital of $3000. May 1, 1850, he took B into partnership, with a capital of $4000. January 1, 1851, they had gained $340. What was each man's share of the gain?

Ans. A's gain $180, B's gain $160.

11. A, B, and C traded in company. A put in $300 for 10 months, B put in $400 for 8 months, and C put in $600 for 2 months. They gained $120. What is the gain of each?

Ans. A's gain 48.64\frac{32}{37}$, B's 51.89\frac{7}{37}$, C's 19.45\frac{35}{37}$.

12. Three men, A, B, and C, hire a pasture in common, for which they are to pay $76.80. A put in 24 oxen for 12 weeks. B put in 25 oxen for 12 weeks, and C put in 30 oxen for 6 weeks. What sum ought each to pay?

Ans. A $28.80, B $30, C $18.

13. John Jones hired a house for one year for $500, with the privilege of admitting two more families if he pleased, with the understanding that all the occupants should have equal privileges in the house. At the end of three months he took in John Smith, and at the end of 9 months Richard Roe. What share of the rent should each pay?

Ans. Jones 291\frac{2}{3}$, Smith 166\frac{2}{3}$, Roe 41\frac{2}{3}$.

14. Two men, A and B, hired a coach in Boston to go to Worcester, the distance being 42 miles, for \$20, with the privilege of taking in two persons more. Having rode 30 miles, they take in C; and on their return from Worcester, when within 20 miles of Boston, they take in D. What ought each man to pay for his accommodation in the coach?

Ans. A \$7.46$\frac{8}{252}$, B \$7.46$\frac{8}{252}$, C \$3.88$\frac{224}{252}$, D \$1.19$\frac{12}{252}$.

15. A and B engage in trade. A puts in a dollars for b months, B puts in c dollars for d months, and they gain e dollars. What share of the gain shall each receive?

Ans. $\frac{abe}{ab+cd}$ A's gain. $\frac{cde}{ab+cd}$ B's gain.

16. A, B, and C engage in trade, with a capital of \$1911. A's money was in the firm 3 months, B's 5 months, and C's 7 months. They gained \$117, which was so divided as that the $\frac{1}{2}$ of A's gain was equal to $\frac{1}{3}$ of B's and $\frac{1}{4}$ of C's gain. What was each man's stock and gain?

Ans. $\left\{\begin{array}{l}\text{A's stock \$693}\frac{273}{2509}\text{, B's \$623}\frac{2002}{2509}\text{, and C's \$594}\frac{234}{2509}\text{.}\\ \text{A's gain \$26, B's gain \$39, and C's gain \$52.}\end{array}\right.$

17. If 12 oxen eat 3$\frac{1}{3}$ acres of grass in 4 weeks, and 21 oxen eat 10 acres in 9 weeks, how many acres would 36 oxen eat in 18 weeks, the grass to be growing uniformly?

Ans. 24 acres.

18. Three men engage in partnership, for 20 months; A, at first, put into the firm \$4000, and at the end of 4 months he put in \$500 more; but, at the end of 16 months, he took out \$1000. B, at first, put in \$3000, but at the end of 10 months he took out \$1500, and at the end of 14 months he put in \$3000. C, at first, put in \$2000, and at the end of 6 months he put in \$2000 more, and at the end of 14 months he put in \$2000 more; but, at the end of 16, he took out \$1500. They had gained, by trade, \$4420. What is each man's share of the gain?

Ans. A's gain, \$1680; B's gain, \$1260; C's gain, \$1480.

SECTION XXVII.

INDETERMINATE ANALYSIS.

ART. 294. In the common rules of Algebra, such questions are usually proposed as require some certain or definite answer; in which case, it is necessary that there should be as many independent equations, expressing their conditions, as there are unknown quantities to be determined; otherwise the problem would not be limited.

But, in other branches of the science, questions frequently arise that involve a greater number of unknown quantities than there are equations to express them; in which instance, they are called *indeterminate*, or unlimited problems, being such as commonly admit of an indefinite number of solutions; although, when the question is proposed in integers, and the answers are required only in whole positive numbers, they are in some cases confined within certain limits, and in others the problem may become impossible.

NOTE.—The rule of Alligation belongs to Indeterminate Analysis. See the Author's NATIONAL ARITHMETIC, page 275.

EXAMPLES.

1. Let $5x+3y=49$.

It is required to solve the equation, and find all the integral and positive values of x and y which are possible.

(1.) By transposition, $3y=49-5x$.

(2.) Dividing as far as possible,

$$y=16-x-\frac{2x-1}{3}.$$

By changing the fraction, for the sake of convenience, to a positive quantity,

$$(3)\qquad y=16-2x+\frac{x+1}{3}.$$

Since we consider only the integral values of y, the fraction must be a whole number.

Let n=that number.

Then $n=\frac{x+1}{3}$.

$3n=x+1.$

$x=3n-1.$

Substituting this value of x in (3),

We have, $y=16-2\ (3n-1)+n.$

Or, $y=18-5n.$

We have now the values of x and y in the terms of n, which must be whole numbers.

By trying various values for n, we shall find all the possible values of x and y.

Let $n=1$, and $x=\ 2$, and $y=13$.
$n=2$, " $x=\ 5$, " $y=8$.
$n=3$, " $x=\ 8$, " $y=3$.
$n=4$, " $x=11$, " $y=-2$.

This last value of y, being negative, is not allowed by the conditions of the question.

The equation, therefore, admits of only three sets of answers.

2. How can $100 be paid with 100 pieces, using eagles, dollars, and "nine-ponces," each of the latter equalling one-eighth of a dollar?

Let x=eagles, y=dollars, z=nine-pences.

(1) Then, $x+y+z=100.$

(2) And $10x+y+\frac{z}{8}=100.$

(3) Multiplying (2) by 8, $80x+8y+z=800.$

(4) Subtracting (1) from (3), $79x+7y=700.$

(5) Transposing, $7y=700-79x.$

(6) Dividing, $y=100-12x+\frac{5x}{7}.$

(7) Let $n=\frac{5x}{7}.$

(8) $7n=5x.$

(9) $x=\frac{7n}{5}.$

(10) By substitution, $y=100-\frac{79n}{5}.$

Let $n=5$, it being the smallest number that will give an integral value to x; and we find $x=7$, and $y=21$, and $z=72$.

Again, let $n=10$, the next smallest number that will make x a positive whole number, and we find $x=14$, and y a negative quantity; and so with every value of n that can be assumed, except 5. The question, then, admits of but one answer; that is, 7 eagles, 21 dollars, and 72 nine-pences.

The answer might have been obtained by eliminating y, instead of z.

(1) Thus, $x+y+z=100.$

(2) And $10x+y+\frac{z}{8}=100.$

(3) Subtracting (1) from (2), $9x-\frac{7z}{8}=0.$

(4) Multiplying, $7z=72x.$

(5) Dividing, $z=10x+\frac{2x}{7}.$

(6) Let $n=\frac{2x}{7}.$

(7) Multiplying, $7n=2x.$

(8) Dividing, $x=\frac{7n}{2}.$

Let $n=2$, it being the least number that will make x a whole number, and $x=7$, and $z=72$, and $y=21$.

If we suppose $n=4$, it being the next larger number that will make x an entire number, then $x=14$, and $z=144$, which is impossible, by the conditions of the question. It is, therefore, certain that no numbers but 7, 21 and 72, are correct.

3. Let $\left.\begin{array}{r} x+\quad y+\quad z=41 \\ \text{And } 24x+19y+10z=741 \end{array}\right\}$ to find all the integral and positive values of x, y, and z.

(1) Conditions, $x+y+z=41.$

(2) And $24x+19y+10z=741.$

(3) Transposing (1), $z=41-x-y.$

(4) Transposing &c. (2), $z=\dfrac{741-24x-19y}{10}.$

(5) Values of (3) and (4), $41-x-y=\dfrac{741-24x-19y}{10}.$

(6) Multiplying, $410-10x-10y=741-24x-19y.$

(7) Reducing, $9y+14x=331.$

(8) Transposing and dividing, $y=\dfrac{331-14x}{9}=36-x+\dfrac{-5x+7}{9}.$

Changing the signs in the last term, so as to make

(9) x positive, $y=36-x-\dfrac{5x-7}{9}.$

(10) Let $\dfrac{5x-7}{9}=n.$

(11) Multiplying, $5x-7=9n.$

(12) Dividing, $x=\dfrac{9n+7}{5}.$

(13) Substituting this for the value of x in the equation (9), we have

$$y=36-\frac{9n+7}{5}-n.$$

(14) Multiplying, $5y=180-9n-7-5n.$

(15) Reducing, $5y=173-14n.$

Let $n=2$, then $y=29$, and $x=5$, and $z=7$.
$n=7$, " $y=15$, " $x=14$, " $z=12$.
$n=12$, " $y=1$, " $x=23$, " $z=17$.

Another solution of the above question:

(1) Let $x+y+z=41.$

(2) And $24x+19y+10z=741.$

(3) Eliminating the x's, we have

$$14z=243-5y.$$

(4) Dividing, $z=17+\frac{5-5y}{14}$.

(5) Let $n=\frac{5-5y}{14}$.

(6) Multiplying, $14n=5-5y$.

(7) Transposing, $5y=5-14n$.

(8) Dividing, $y=1-\frac{14n}{5}$.

(9) Reducing, &c., $y=1-3n+\frac{n}{5}$.

We might use the first value of y; but, to do what it is convenient to do in some cases, let us introduce a second auxiliary quantity, to represent the fraction in the 2d value of y.

(10) Let $m=\frac{n}{5}$.

(11) Multiplying, $5m=n$.

(12) By substitution, $y=1-14m$.

(13) And $z=17+\frac{5-(5-70m)}{14}$.

(14) Therefore, $z=17+5m$.

The value of y requires that m should be zero, or negative.

Let us first suppose the value of m to be 0.

Then, $y=1-14(0)=1$.

And $z=17-5(0)=17$.

And $x=41-1-17=23$.

Let -1 be taken for the value of m.

And $y=1-(-14)=1+14=15$.

" $z=17+5(-1)=17-5=12$.

" $x=41-12-15=14$.

Again, let -2 be taken for the value of m.

And $y=1-14(-2)=29$.

And $z=17+5(-2)=7.$

" $x=41-29-7=5.$

Again, let -3 be taken for the value of m.

Then, $y=1-14(-3)=43.$

This value of y is more than the united values of x, y, and z by the conditions of the question.

The three values of m ($0, -1, -2$), then, are the only ones which will give integral and positive values for all the quantities.

The reason for using the second quantity (m) was to avoid fractions in the values of y and z. Three or four successive auxiliary quantities may be used advantageously in some cases.

295. To find two square numbers whose sum shall be a square.

4. Let $x^2+y^2=z^2.$

Then $x^2=z^2-y^2=(z+y)(z-y).$

Multiplying both sides by m, we have $mx^2=m(z+y)(z-y).$

Assuming, $mx=z+y$, and $x=m(z-y)$,

We have, $z+y=m^2(z-y).$

Therefore, $(m^2+1)y=(m^2-1)z=(m^2-1)(mx-y)=$
$(m^2-1)mx-(m^2-1)y.$

Therefore, $2m^2y=(m^2-1)mx.$

And $$x=\frac{2my}{m^2-1}.$$

To obtain whole numbers without fractions, let $y=m^2-1$; then we have $x=2m$, and $z=m^2+1$. That is, the general *forms* of the three numbers will be $x=2m$, $y=m^2-1$, and $z=m^2+1$.

If $m=1$, we have $x=2$, $y=0$, and $z=2$.
$m=2$, " $x=4$, $y=3$, " $z=5$.
$m=3$, " $x=6$, $y=8$, " $z=10$.
$m=4$, " $x=8$, $y=15$, " $z=17$.
$m=5$, " $x=10$, $y=24$, " $z=26$.

The pupil will perceive that the values of x and y may represent the base and perpendicular of a right-angled triangle, and z the hypothenuse.

296. To find two numbers the sum of whose squares is given.

By substitution, we have

$$x=\left(\frac{2m}{m^2+1}\right)z, \text{ and } y=\left(\frac{m^2-1}{m^2+1}\right)z.$$

5. Find the values of x and y which will satisfy the equation $x^2+y^2=10^2$.

Here $$x=\left(\frac{2m}{m^2+1}\right)10, \text{ and } y=\left(\frac{m^2-1}{m^2+1}\right)10.$$

In these equations, any number may be assigned for the value of m.

If $m=1$, we have $x=10$, and $y=0$.
$m=2$, " $x=8$, " $y=6$.
$m=3$, " $x=6$, " $y=8$.
$m=4$, " $x=\frac{80}{17}$, " $y=\frac{150}{17}$.
$m=8$, " $x=\frac{32}{13}$, " $y=\frac{126}{13}$, &c.

297. To find two square numbers whose difference shall be a square number.

6. Let $x^2-y^2=z^2$; therefore $(x+y)m(x-y)=mz^2$; whence, assuming $x+y=mz$, and $m(x-y)=z$, we have $x+y=m^2(x-y)$, and $(m^2+1)y=(m^2-1)x$.

Therefore, $x=\left(\frac{m^2+1}{m^2-1}\right)y$, and if $y=m^2-1$, then will $x=m^2+1$, and $z=2m$.

If $m=1$, we have $x=2$, $y=0$, and $z=2$.
$m=2$, " $x=5$, $y=3$, " $z=4$.
$m=3$, " $x=10$, $y=8$, " $z=6$.
$m=4$. " $x=17$, $y=15$, " $z=8$, &c.

We might assume a fractional value for m.

298. If the difference of the two squares be given, we have the following formula for ascertaining their value;

$$m(x+y)=m^2z, \text{ and } m(x-y)=z.$$

Whence, $2mx=(m^2+1)z$, $x=\left(\frac{m^2+1}{2m}\right)z$, and $y=\left(\frac{m^2-1}{2m}\right)z$

7. What values of x and y will satisfy the equation $x^2-y^2=24^2$?

Here $\qquad x=\left(\frac{m^2+1}{2m}\right)24$, and $y=\left(\frac{m^2-1}{2m}\right)24$,

where the values of m may be assumed at pleasure.

If $m=1$, we have $x=24$, and $y=0$.
$m=2$, " $x=30$, " $y=18$.
$m=3$, " $x=40$. " $y=32$.
$m=4$, " $x=51$, " $y=45$, &c.

8. The difference between the squares of the ages of two persons at one period was 45, and at another it was 159. Required the age of each.

Ans. At the first period their ages were 9 and 6, and at the second 28 and 25. Or, at the first period they were 23 and 22, and at the second 80 and 79.

EXAMPLES.

1. How many pounds of sugar, at 11 cents per lb., shall be mixed with another kind, at 5 cents per lb., that the mixture shall be worth $2.54?

Ans. 19 lbs. with 9 lbs.; 14 lbs. with 20 lbs.; 9 lbs. with 31 lbs.; and 4 lbs. with 42 lbs.

2. A person divides 65 shillings among 15 persons, men, women, and children. The share of a man is 7 shillings, that of a woman 3 shillings, and that of a child 2 shillings. How many persons were there of each class?

Ans. 6 men, 5 women, and 4 children.

3. A gentleman has two farms, valued at $2000. The best is worth $21 per acre, and the other $17 per acre. How many acres are there in each farm?

Ans. The first may contain 92, 75, 58, 41, 24, or 7 acres; and the second may contain 4, 25, 46, 67, 88, or 109 acres.

4. I purchase wheat at 17 shillings and barley at 11 shillings a bushel, and expend in all £27 2*s*. How many bushels of each do I purchase? *Ans.* 6 of wheat and 40 of barley; or 17 of wheat and 23 of barley; or 28 of wheat and 6 of barley.

5. It is required to divide 100 into two such parts that one of them may be divisible by 7, and the other by 11.

Ans. The only parts are 56 and 44.

6. In how many ways can a debt of $25 be paid with $2 and $3 bills? *Ans.* Four ways.

7. I wish to mix corn at 70 cents per bushel with wheat at $1.90 per bushel. How many bushels of each must be taken to amount to $9.20? *Ans.* 5 bushels of corn, and 3 of wheat.

8. It is required to find the least whole number which, being divided by 17, shall leave a remainder of 7, and, when divided by 26, shall leave a remainder of 13. *Ans.* 143.

9. A person wishes to purchase 20 animals for £20; sheep at 31 shillings, pigs at 11 shillings, and rabbits at 1 shilling each. In how many ways can he do it?

Ans. He can buy 12 sheep, 2 pigs, 6 rabbits; or 11 sheep, 5 pigs, 4 rabbits; or 10 sheep, 8 pigs, 2 rabbits.

NOTE. — The question will admit of only these three answers.

10. It is required to find two numbers, one of which being multiplied by 7, and the other by 13, the sum of the products shall be equal to 71.

NOTE. — This question does not admit of an answer in whole numbers. No value can be given to the auxiliary unknown quantity (n), which will render x and y both integral and positive.

11. It is required to find two numbers the sum of whose squares shall be 1225.

Ans. The only positive and integral numbers are 21 and 28.

12. The difference of the squares of two numbers is 1521; what are the numbers? *Ans.* 52 and 65, or &c. &c.

SECTION XXVIII.

VARIATIONS, PERMUTATIONS, AND COMBINATIONS.

Art. 299. The different arrangements that can be made of any number of quantities, taking a certain number at a time, are called *Variations.*

Thus, if a, b, c, be taken two together, the variations will be ab, ba, ac, ca, bc, cb.

And if a, b, c, d, be taken three together, their variations will be 24. Thus,

abc abd acb acd adb adc
bac bad bca bcd bda bdc
cab cad cba cbd cda cdb
dab dac dba dbc dca dcb.

If all the quantities are taken together, their variations are called *Permutations.*

Thus the permutations of a, b, c, are abc, acb, bac, bca, cab, cba.

The permutations of 1, 2, 3, are 123, 132, 213, 231, 312, 321.

The different collections that can be made of a number of things, taking a certain number of things together without regarding their order, are called *Combinations.* Thus the combinations of a, b, c, taken two together, are ab, ac, bc.

Each combination will supply as many corresponding variations as the number of things it contains admits of permutations.

VARIATIONS.

Let $V =$ the number of variations required.
$n =$ number of different things.
$r =$ number of things taken.

The following, therefore, will be the formula for obtaining the number of variations of n things, taken r together.

The number of variations of n things, taken r together, is $n(n-1)(n-2)\ldots.[n-(r-1)]$.

Let a, b, c, d, &c., be the n things; then the number of variations which can be made, taking them singly, is n.

Let $n-1$ of these things, namely, b, c, d, &c., be taken singly; then the number of their variations is $n-1$; and, if a be placed before each, we shall have $n-1$ variations of n things, taken two together, in which a stands first. Similarly, we shall have $n-1$ such variations in which b stands first, and similarly for all the n things; hence there will be, on the whole, $(n-1)$ variations of n things, taken *two* together.

Again, taking $n-1$ of these things, namely, b, c, d, &c., their variations, taken two together, will be $n(n-1)(n-2)$; and proceeding as before, there will be, in the whole, $(n-1)(n-2)$ variations of n things, taken *three* together.

Similarly, their variations, taken four together, will be $n(n-1)(n-2)(n-3)$. Hence, if V_1, V_2, V_3, &c., V_r, denote the variations of n things, taken 1, 2, 3, &c., r, together, we have

$$V_1=n,\ V_2=n(n-1),\ V_3=n(n-1)(n-2),\ \&c.$$

$$V_r=n(n-1)(n-2)(n-3)\ldots..[n-(r-1)].$$

From the above we infer that the permutations (p) of n things are their variations taken *all together*; therefore, by writing n for r, we shall have

$$p=n(n-1)(n-2)\ldots..(n-(n-2))(n-(n-1))=$$

$$n(n-1)(n-2)\ldots.2.1=1.2.3\ldots.n.$$

1. How many changes can be rung with 7 bells out of 10?

$$V=n(n-1)(n-2)\ldots.(n-(r-1)).$$

As there are 10 bells, $n=10$; and as they are taken 7 at a time, $r=7$, and $r-1=6$; therefore, $n-(r-1)=10-6=4$.

Hence $V_7=10.9.8.7.6.5.4=604800$ changes. *Ans.*

2. How many words can be made with 4 letters out of 5?

Ans. 120.

3. How often can 4 boys change their places in a class of 8 so as not to preserve the same order? *Ans.* 1680.

PERMUTATIONS.

300. When a and b are different, their permutations are ab, ba; but, when $a=b$, they become aa.

Let a recur p times; b, q times; c, r times; and P be the number of permutations required. Then, if all the a's be changed into different letters, they will form $1.\ 2.\ 3.\ldots\ldots p$, permutations; and, out of *each* of the P permutations, we should form $1.\ 2.\ 3.\ldots\ldots$ permutations. In like manner, if all the b's were changed to different letters, they would form $1.\ 2.\ 3\ldots\ldots q$ permutations; and, therefore, there would be $P.\ (1.\ 2.\ 3.\ldots\ldots p.\ 1.\ 2.\ 3.\ldots\ldots q)$ permutations. Now, when all the quantities have become different, the number of permutations is $1.\ 2.\ 3.\ 4.\ldots\ldots n$. by Art. 299.

Therefore, $P.\ (1.\ 2.\ 3.\ldots p.\ 1.\ 2.\ 3.\ldots q.\ 1.\ 2.\ 3.\ldots r.\ \&c.)$

$$=1.\ 2.\ 3.\ldots n.$$

Whence, $$P=\frac{1.\ 2.\ 3.\ldots n}{1.\ 2.\ 3.\ldots p.\ 1.\ 2.\ 3.\ldots q.\ 1.\ 2.\ 3.\ldots r,\ \&c.}.$$

4. In how many ways may the word *enunciation* be written?

In this word there are 11 letters, of which 3 are n's and 2 are i's; therefore, $n=11$, $p=3$, $q=2$.

Hence, $$P=\frac{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7.\ 8.\ 9.\ 10.\ 11}{1.\ 2.\ 3.\ 1.\ 2}$$

$=3326400$ ways. *Ans.*

5. In how many ways may the word *algebra* be written?
Ans. 2520.

6. How many different numbers can be made with the following figures, 1225555? *Ans.* 105.

7. How many variations may be made of the letters in the word *zaphnathpaaneah*? *Ans.* 454053600.

COMBINATIONS.

301. The different collections that can be made of a number of things, taking a certain number together, without regarding their order, are called their *Combinations.*

Thus, the combinations of *a*, *b*, *c*, taken two together, are *ab*, *ac*, *bc*.

Each combination will supply as many corresponding variations as the number of things it contains admits of permutations.

Each combination of r things supplies $1.\ 2.\ 3.\ \ldots\ldots r$ variations of r things; hence, if C_r be the number of combinations of n things, taking r together, the following will be the formula.

$$C_r\ (1.\ 2.\ 3.\ \ldots\ r)=V_r=n(n-1)(n-2)\ \ldots\ldots\ (n-(r-1)).$$

$$\text{Therefore, } C_r=\frac{n(n-1)(n-2)\ \ldots\ldots\ (n-(r-1))}{1.\ 2.\ 3.\ \ldots\ r}.$$

8. Into how many different triangles may a decagon be divided, by drawing lines from the angular points?

Note. — The number of triangles will be equal to the number of lines that can be drawn by connecting 7 at a time of the 10 angles, with each angle; taken 7 together,

$$C=\frac{n(n-1)(n-2)\ \ldots\ldots\ (n-(r-1))}{1.\ 2.\ 3.\ \ldots\ldots\ r}=\frac{10.\ 9.\ 8.\ 7.\ 6.\ 5.\ 4}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7}$$
$$=120.\quad \textit{Ans.}$$

9. How many different combinations can be made with 5 letters out of 8? *Ans.* 56.

10. From a company of 12 persons, it is proposed to ascertain how many parties, of ten each, can be selected, and no two parties to be composed of the same individuals. How many parties can be selected? *Ans.* 66.

11. A company of soldiers consists of 40 men, and 6 of them are selected every night to mount guard; on how many nights can a different guard of 6 sentinels be made? *Ans.* 3838380.

12. How many different numbers can be made out of one unit, two 2's, three 3's, and four 4's, supposing all the figures to be in every number? *Ans.* 12600.

13. What is the total number of combinations of 16 things, taken 1, 2, 3, &c., at a time? *Ans.* 65535.

SECTION XXIX.

LOGARITHMS.*

Art. 302. Logarithms are a series of numbers in arithmetical progression, answering to another series of numbers in geometrical progression.

$$\text{Thus,}\ \left\{\begin{array}{l} 0,\ 1,\ 2,\ 3,\ \ 4,\ \ 5,\ \ 6,\ \text{indices, or logarithms.} \\ 1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ \text{geometrical progression.} \end{array}\right.$$

$$\text{Or,}\ \left\{\begin{array}{l} 0,\ 1,\ 2,\ \ 3,\ \ 4,\ \ \ 5,\ \ \ 6,\ \text{indices, or logarithms.} \\ 1,\ 3,\ 9,\ 27,\ 81,\ 243,\ 729,\ \text{geometrical progression.} \end{array}\right.$$

$$\text{Or,}\ \left\{\begin{array}{l} 0,\ \ 1,\ \ \ 2,\ \ \ \ 3,\ \ \ \ \ \ 4,\ \ \ \ \ \ \ 5,\ \text{indices, or log.} \\ 1,\ 10,\ 100,\ 1000,\ 10000,\ 100000,\ \text{geomet. prog.} \end{array}\right.$$

From the above, it is evident that the same indices may serve equally for any geometrical series; and, consequently, there may be an endless variety of systems of logarithms to the same common numbers, by only changing the second term, 2, 3, or 10, &c., of the geometrical series of whole numbers; and, by interpolation, the whole system of numbers may be made to enter the geometrical series, and receive their proportional logarithms, whether integers or decimals.

It is also apparent, from the nature of these series, that, if any two indices be added together, their sum will be the index of that number which is equal to the product of the two terms in the geometrical progression to which those indices belong. Thus the indices 2 and 3, being taken together, make 5; and the numbers 4 and 8, or the terms corresponding to those indices, being multiplied together, make 32, which is the number answering to the index 5.

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* The invention of Logarithms is due to Lord Napier, Baron of Merchiston, in Scotland, and is properly considered as one of the most useful inventions of modern times. A table of these numbers was first published by the inventor at Edinburgh, in the year 1614, in a treatise entitled *Canon Mirificum Logarithmorum*, which was eagerly read by all the learned throughout Europe. Mr. Henry Briggs, then professor of geometry at Gresham College, soon after the discovery went to visit the noble inventor; after which, they jointly undertook the arduous task of computing new tables on this subject, and reducing them to a more convenient form than that which was at first thought of. But, Lord Napier dying soon after, the whole burden fell upon Mr. Briggs; who, with prodigious labor and great skill, made an entire canon, according to the new form, for all numbers, from 1 to 20000, and from 90000 to 101000, to 14 places of decimals, and published it in London, in the year 1624.

In like manner, if any one index be subtracted from another, the difference will be the index of that number, which is equal to the quotient of the two terms to which those indices belong. Thus the index 6, minus the index 4, is 2; and the terms corresponding to those indices are 64 and 16, whose quotient is 4, which is the number answering to the index 2.

For the same reason, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power. Thus, the index or logarithm of 4, in the above series, is 2; and, if this number be multiplied by 3, the product will be 6, which is the logarithm of 64, or the third power of 4.

And, if the logarithm of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root. Thus, the index or logarithm of 64 is 6; and, if this number be divided by 2, the quotient will be 3, which is the logarithm of 8, or the square root of 64.

The logarithms most convenient for practice are such as are adapted to a geometrical series increasing in a ten-fold ratio, as in the last of the above forms; and are those which are to be found, at present, in most of the common tables on this subject. The distinguishing mark of this system of logarithms is, that the index or logarithm of 10 is 1; that of 100, 2; that of 1000, 3, &c.

In decimals, the logarithm of .1 is —1, and that of .01 is —2, that of .001 is —3, and so on. The logarithm of 1 in every system being 0, it follows that the logarithm of any number between 1 and 10 must be 0 and some fractional parts, and that of a number between 10 and 100 will be 1 and some fractional part, and so on for any other number whatever. And, since the integral part of a logarithm, usually called the Index or Characteristic, is always thus readily found, it is commonly omitted in the tables; being left to be supplied by the operator himself, as occasion requires.

303. Another definition of Logarithms is, that the logarithm is the index of that power of some other number which is equal to the given number. So, if there be $N=r^n$, then n is the logarithm of N; where n may be either positive or negative, or nothing, and the root, r, any number whatever, according to the different systems of logarithms.

When n is $= 0$, then N is $= 1$, whatever the value of r is, which shows that the logarithm of 1 is always 0 in every system of logarithms. When $n = 1$, then $N = r$; so that the radix, r, is always that number whose logarithm is 1, in every system. When the radix $r = 2.718281828459$, &c., the indices n are the hyperbolic, or Napier's logarithm of numbers, N; so that n is always the hyperbolic logarithm of the number N, or $(2.718281828459)^n$.

304. When the radix $r = 10$, then the index n becomes the common or Briggs' logarithm of the number N; so that the common logarithm of any number 10^n or N is n, the index of that power of 10 which is equal to the said number. Thus, 100, being the second power of 10, will have 2 for its logarithm; and 1000, being the third power of 10, will have 3 for its logarithm. Hence, also, if $50 = 10^{1.69897}$, then is 1.69897 the common logarithm of 50. That is, 10 has been raised to the 169897th power, and the 100000d root has been extracted, which is found to be 50, nearly. And, in general, the following decuple series of terms, namely,

$$\begin{array}{lccccccccc} & 10^4, & 10^3, & 10^2, & 10^1, & 10^0, & 10^{-1}, & 10^{-2}, & 10^{-3}, & 10^{-4}, \\ \text{or} & 10000, & 1000, & 100, & 10, & 1, & .1, & .01, & .001, & .0001, \\ \text{have} & 4, & 3, & 2, & 1, & 0, & -1, & -2, & -3, & -4, \end{array}$$

for their logarithms, respectively. And from this scale of numbers and logarithms the same properties easily follow, as above mentioned.

305. To compute the Logarithm to any of the Natural Numbers, 1, 2, 3, 4, 5, &c., we have the following

Rule. *Take the geometrical series,* 1, 10, 100, 1000, 10000, *&c., and apply it to the arithmetical series,* 0, 1, 2, 3, 4, 5, *&c., as logarithms.*

Find a geometrical mean between 1 *and* 10, *or between* 10 *and* 100, *or any other two adjacent terms of the series, between which the number proposed lies.*

In like manner, between the mean thus found, and the nearest extreme, find another geometrical mean; and so on, till you arrive within the proposed limit of the number whose number is sought.

Find, also, as many arithmetical means in the same as you found geometrical ones, and these will be the logarithms answering to the said geometrical means.

EXAMPLE.

Calculate the logarithm of 9.

Here the proposed number lies between 1 and 10.

First, then, the log. 10 is 1, and the log. of 1 is 0.

Therefore $(1+0)\div2=\frac{1}{2}=.5$ is the arithmetical mean.

And $(10\times1)^{\frac{1}{2}}=3.1622777$, the geometrical mean.

Hence the log. of 3.1622777 is .5.

Secondly, the log. of 10 is 1, and the log of 3.1622777 is .5.

Therefore $(1+.5)\div2=.75$, the arithmetical mean.

And $(10\times3.1622777)^{\frac{1}{2}}=5.6234132$, the geometrical mean.

Hence the log. of 5.6234132 is .75.

Thirdly, the log. of 10 is 1, and the log. of 5.6234132 is .75.

Therefore $(1+.75)\div2=.875$ is the arithmetical mean.

And $(10\times5.6234132)^{\frac{1}{2}}=7.4989422$ the geometrical mean.

Hence the log. of 7.4989422 is .875.

Fourthly, the log. of 10 is 1, and the log. of 7.4989422 is .875.

Therefore, $(1+.875)\div2=.9375$ is the arithmetical mean.

And $(10\times7.4989422)^{\frac{1}{2}}=8.6596431$, the geometrical mean.

Hence the log. of 8.6596431 is .9375.

Fifthly, the log. of 10 is 1, and the log. of 8.6596431 is .9375.

Therefore, $(1+.9375)\div2=.96875$ is the arithmetical mean.

And $(10\times8.6596431)^{\frac{1}{2}}=9.3057204$, the geometrical mean.

Hence the log. of 9.3057204 is .96875.

Sixthly, the log. of 8.6596431 is .9375, and the log. of 9.3057204 is .96875.

Therefore, $(.9375+.96875)\div2=.953125$ is the arithmetical mean.

And $(8.6596431\times9.3057204)^{\frac{1}{2}}=8.9768713$, the geometrical mean.

Hence the log. of 8.9768713 is .953125.

By proceeding in this manner, after 25 extractions, it will be found that the logarithm of 8.9999998 is .9542425, which may be taken for the logarithm of 9, as it differs so little from it, and is sufficiently exact for all practical purposes; and in this manner were the logarithms of almost all the prime numbers at first computed.

30[illegible]. Another method of computing logarithms is by the aid of a given decimal.

RULE. *Let* b *be the number whose logarithm is required to be found, and* a *the number next less than* b, *so that* $b-a=1$, *the logarithm of* a *being known; and let* s *denote the sum of the two numbers,* $a+b$. *Then*

1. *Divide the constant decimal* .8685889638 *by* s, *and reserve the quotient; divide the reserved quotient by the square of* s, *and reserve this quotient; divide this last quotient, also, by the square of* s, *and again reserve the quotient; and thus proceed, continually dividing the last quotient by the square of* s, *as long as division can be made.*

2. *Write these quotients orderly, under one another, the first uppermost, and divide them respectively by the odd numbers,* 1, 3, 5, 7, 9, *&c., as long as division can be made; that is, divide the reserved quotient by* 1, *the second by* 3, *the third by* 5, *the fourth by* 7, *and so on.*

3. *Add all these last quotients together, and the sum will be the logarithm of* b÷a. *To this logarithm add, also, the given logarithm of the said next less number,* a; *the last sum will be the logarithm of the number* b *proposed.*

EXAMPLES.

1. Let it be required to find the logarithm of the number 2.

Here the given number b is 2, and the next less number a is 1, whose logarithm is 0; also, the sum $2+1=3=s$, and its square $s^2=9$. Then the operation will be as follows.

3)	868588964	1)	289529654	(	289529654
9)	289529654	3)	32169962	(	10723321
9)	32169962	5)	3574440	(	714888
9)	3574440	7)	397160	(	56737
9)	397160	9)	44129	(	4903
9)	44129	11)	4903	(	446
9)	4903	13)	545	(	42
9)	545	15)	61	(	4
9)	61				

Logarithm of $\frac{2}{1}$=.301029995
Add logarithm of 1=.000000000

Logarithm of 2=.301029995

2. Compute the logarithm of the number 3.

Here $b=3$, the next less number $a=2$, and the sum $a+b=5=s$, whose square $s^2=25$.

5)	.868588964	1)	.173717793	(	.173717793
25)	.173717793	3)	6948712	(	2316237
25)	6948712	5)	277948	(	55590
25)	277948	7)	11118	(	1588
25)	11118	9)	445	(	50
25)	445	11)	18	(	2
	18				

Logarithm of $\frac{3}{2}$=.176091260
Logarithm of 2 add.=.301029995

Logarithm of 3=.477121255

307. Because the sum of the logarithms of numbers gives the logarithm of their product, and the difference of the logarithms gives the logarithm of the quotient of the number, we may, therefore, from the above two logarithms, and the logarithm of 10, which is 1, raise a great many logarithms, as will appear by the following

EXAMPLES.

1. To find the logarithm of 4, we multiply the logarithm of 2=.301030 by 2, because twice 2 are 4.

Logarithm of 2=.301030
2
Logarithm of 4=.602060

2. Find the logarithm of 6.

Because $2\times3=6$, we add their logarithms.

Logarithm of 2=.301030
Logarithm of 3=.477121
Logarithm of 6=.778151

3. Find the logarithm of 8.

Because $2^3=8$, therefore

Logarithm of 2=.301030
Multiplied by 3= 3
Gives logarithm of 8=.903090

4. Find the logarithm of 9.

Because $3^2=9$, therefore

Logarithm of 3=.477121
Multiplied by 2= 2
Gives logarithm of 9=.954242

5. Find the logarithm of 5.

Because $\frac{10}{2}=5$, therefore

From logarithm of 10=1.000000
Subtract logarithm of 2= .301030
Logarithm of 5. *Ans.* .698970

Having computed by the general rule the logarithms of the other prime numbers, 7, 11, 13, 17, 19, 23, &c., then, by composition and division, we may easily find as many logarithms as we please.

NOTE.—The index of every logarithm is always *one less* than the integers to the given number.

308. To find in the table the logarithm of any number.

(1.) If the given number be less than 100, or consist of only two figures.

RULE. *Enter the first page of the table, which contains all the numbers from* 1 *to* 100, *and opposite the given number will be found the logarithm with the index prefixed.*

(2.) If the given number be more than 100, and less than 1000.

RULE. *Find the given number in the left-hand column of the table, and opposite, in the next column, will be found the logarithm to which the index,* 2, *must be prefixed.*

Thus, if the logarithm of 189 were required, we find this number in the table, and, opposite to it, we find the logarithm .276462. To this we prefix the index, 2, and we have 2.276462.

(3.) If the given number be more than 1000, and less than 10000.

RULE. *Find the first three figures of the given number in the left-hand column, and, opposite to it, in the column marked at the top with the fourth figure, is the logarithm required. To which must be prefixed the index,* 3.

Thus, if the logarithm of 3568 were required, we find opposite 356, in the left-hand column, and under 8, found at the top of the column, .552425. To this we prefix the index, 3, because there are four figures in the given number, thus, 3.552425.

(4.) If the given number be more than 10000.

RULE. *Find the logarithm of the first four figures as before, also the next greater logarithm; subtract the one logarithm from the other, as also their corresponding numbers, the one* *from the other. Then say, As the difference between the two numbers is to the difference of their logarithms, so is the remaining part of the given number to the proportional part of the logarithm; which part, being added to the less logarithm before taken out, gives the whole logarithm nearly.*

EXAMPLES.

1. Find the logarithm of 340926.

The logarithm of	340900	is	$=.532627$
The logarithm of	341000	is	$=.532754$
The differences are $=$	100		127

Then, as $100 : 127 :: 26 : 33$, the proportional part. This added to the first logarithm $(.532627+33)$ gives .532660. To this we prefix the index 5, because the given number had six figures.

(5.) To find the logarithm of a number consisting of an integer and decimal.

Rule. *Find the logarithm of the decimal part the same as if all its figures were integral; then this, having prefixed to it the proper index, will give the logarithm required; remembering that the index will always be* one less *than the integer.*

Thus the logarithm of 42.25 is 1.625827.

(6.) To find the logarithm of a proper fraction.

Rule. *Subtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm sought; which, being that of a decimal fraction, must always have a negative index.*

2. What is the logarithm of $\frac{37}{94}$?

Logarithm of 37	$=1.568202$
Logarithm of 94	$=1.973128$
Logarithm of $\frac{37}{94}$	$=-1.595074$

(7.) To find the logarithm of a mixed number.

Rule. *Reduce the mixed number to an improper fraction, and find the difference of the logarithms of the numerator and denominator in the same manner as above.*

3. What is the logarithm of $17\frac{14}{23}$?

First $17\frac{14}{23}=\frac{405}{23}$. Then,
Logarithm of 405 =2.607455
Logarithm of 23 =1.361728

Logarithm of $17\frac{14}{23}$ =1.245727

(8.) To find the logarithm of any decimal.

Rule. *Find the logarithm of the decimal as of an integer, and if the first significant figure in the decimal occupy the place of tenths, the index will be* —1. *Thus the logarithm of* .375 *will be* —1.574031. *If the first decimal place occupy the place of hundredths, the index will be* —2. *If the decimal is preceded by two ciphers, the index will be* —3, *and so on.*

Thus the logarithm of .234 = —1.369216
of .0234 = —2.369216
of .00234 = —3.369216
of .000234 = —4.369216
of .0000234 = —5.369216

EXAMPLES.

1. What is the logarithm of 1728? *Ans.* 3.237544.
2. What is the logarithm of 23.56? *Ans.* 1.372175.
3. What is the logarithm of 89632? *Ans.* 4.952462.
4. What is the logarithm of $\frac{17}{93}$? *Ans.* —1.261966.
5. What is the logarithm of $\frac{3}{107}$? *Ans.* —2.447737.
6. What is the logarithm of $19\frac{2}{37}$? *Ans.* 1.279987.
7. What is the logarithm of .3076? *Ans.* —1.487986.
8. What is the logarithm of .00016? *Ans.* —4.204120.
9. What is the logarithm of .0000006? *Ans.* —7.778151.

309. To find the natural number to any given logarithm.

This is to be found in the tables by the reverse method to the former, by searching for the proposed logarithm among those in the table, and taking out the corresponding number by inspection, in which the proper number of integers is to be pointed off, that is, one more than the index. For, in finding the number answering to any given logarithm, the index always shows how far the first figure must be removed from the place of units to the left hand, or integers, when the index is affirmative, but the right hand, or decimals, when it is negative.

Thus the number to the logarithm 1.532882 is 34.11.

And the number of the logarithm —1.532882 is .3411.

But, if the logarithm cannot be exactly found in the table, we adopt the following

Rule. *Take out the next greater and the next less, subtracting one of these logarithms from the other, as also their natural numbers the one from the other, and the less logarithm from the logarithm proposed. Then say, As the difference of the first, or tabular logarithms, is to the difference of their natural numbers, so is the difference of the given logarithm and the least tabular logarithm to the corresponding numeral difference; which, being annexed to the least natural number above taken, gives the natural number sought, corresponding to the proposed logarithm.*

EXAMPLE

1. What is the natural number answering to the given logarithm 1.532708?

Next greater,	532754; its number,	341000; given log.,	532708
Next less,	532627; its number,	340900; next less,	532627
	127	100	81

Then, as 127 : 100 : : 81 : 64, nearly the numeral difference. Therefore, 340900 +64=34.0964, marking off two integers, because the index of the given logarithm is 1.

Had the index been —1.532708, its corresponding number would have been .340964, wholly a decimal.

MULTIPLICATION OF LOGARITHMS.

Rule. *Take out the logarithms of the factors from the table, then add them together, and their sum will be the logarithm of the product required. Then take out from the table the natural number answering to the sum for the product sought. Add what is to be carried from the decimal part of the logarithm to the affirmative index or indices, or else subtract it from the negative. Also, adding the indices together, when they are of the same kind, both affirmative or both negative; but subtracting the less from the greater when the one is affirmative and the other negative, and prefixing the sign of the greater to the remainder.*

EXAMPLES.

1. Multiply 23.14 by 5.062.

	Numbers.		Logarithms.
	23.14	=	1.364363
	5.062	=	0.704322
Product,	117.1343	=	2.068685

2. Multiply 2.581926 by 3.457291.

	Numbers.		Logarithms.
	2.581926	=	0.411944
	3.457291	=	0.538736
Product,	8.92647	=	0.950680

3. What is the continued product of 3.902, 597.16, and .0314728?

	Numbers.		Logarithms.
	3.902	=	0.591287
	597.16	=	2.776091
	.0314728	=	—2.497935
Product,	73.335	=	1.865313

Here the —2 cancels the +2, and the 1 to carry from the decimal is set down.

5. What is the continued product of 3.586, 2.1046, 0.8372, and 0.0294?

Numbers. Logarithms.
3.586 = 0.554610
2.1046 = 0.323170
0.8372 = —1.922829
0.0294 = —2.468347

Product, 0.1857615 = —1.268956

Here the 2 to carry cancels the —2, and there remains —1 to set down.

DIVISION BY LOGARITHMS.

Rule. *From the logarithm of the dividend subtract the logarithm of the divisor, and the number answering to the remainder will be the quotient required. Change the sign of the index of the divisor from affirmative to negative, or from negative to affirmative; then take the sum of the indices, if they be of the same name, or their difference, when of different signs, with the sign of the greater, for the index to the logarithm of the quotient. And also, when 1 is borrowed in the left-hand place of the decimal part of the logarithm, add it to the index of the divisor when that index is affirmative, but subtract it when negative; then let the sign of the index arising from hence be changed, and worked with as before.*

EXAMPLES.

1. Divide 24163 by 4567.

Logarithm of 24163 = 4.383151
Logarithm of 4567 = 3.659631

Quotient, 5.29078 = 0.723520

2. Divide 37.149 by 523.76.

Logarithm of 37.149 = 1.569947
Logarithm of 523.76 = 2.719132

Quotient, .0709275 = —2.850815

3. Divide .06314 by .007241.

Logarithm of .06314 = —2.800305
Logarithm of .007241 = —3.859799

Quotient, 8.71978 = 0.940506

Here 1 carried from the decimals to the —3 makes it become —2, which, taken from the other —2, leaves 0 remainder.

4. Divide .7438 by 12.9476.

Logarithm of .7438 = —1.871456
Logarithm of 12.9476 = 1.112189

Quotient, .057447 = —2.759267

Here 1 taken from the —1 makes it become —2 to set down.

310. To find the Arithmetical Complement of the logarithm of any number.

Rule. *Subtract the logarithm of the number from the logarithm of* 1, *which is zero* (0).

EXAMPLES.

1. What is the arithmetical complement of 1.462398?

0.
1.462398

—2.537602

2. What is the arithmetical complement of —1.397940?

0.
—1.397940

0.602060

3. What is the arithmetical complement of —3.678914?

0.
—3.678914

2.321086

4. What is the arithmetical complement of 3.614582?

0.
3.614582

—4.385418

5. What is the arithmetical complement of -4.321617? *Ans.* 3.678383.

6. What is the arithmetical complement of 0.781562? *Ans.* -1.218438.

7. What is the arithmetical complement of 5.321463? *Ans.* -6.678537.

8. What is the arithmetical complement of 3.456321? *Ans.* -4.543679.

The pupil will understand the rationale of this rule, by observing that the product of a, multiplied by b, is the same as a divided by $\frac{1}{b}$.

Thus, $a\times b=ab$, or $a\div\frac{1}{b}=ab$.

Or, 12 multiplied by 5 is the same as 12 divided by $\frac{1}{5}$.

Thus, $12\times5=60$; or $12\div\frac{1}{5}=60$.

The same by logarithms.

Logarithm of 12, $=1.079181$
Logarithm of 5, $=0.698970$

Logarithm of the product, $60=1.778151$.

Or,

Logarithm of 12, $=1.079181$
Logarithm of $\frac{1}{5}=.2=-1.301030$ Arith. Com. $=0.698970$

Logarithm of the product, 60, $=1.778151$.

311. Any number may be divided by adding the arithmetical complement of the divisor to the logarithm of the dividend. Their sum will give the logarithm of the quotient.

9. Divide 1728 by 12.

Logarithm of 1728, $=3.237544$
Logarithm of $12=1.079181$ Arith. Com. $=-2.920819$

Ans. $144=2.158363$

10. What is the value of x in the following equation?

$$x=\frac{1728\times144\times6}{36\times18\times12}.$$

Log. 1728 = 3.237544
Log. 144 = 2.158362
Log. 6 = 0.778151
Log. 36=1.556303 Arith. Com. =—2.443697
Log. 18=1.255273 " " =—2.744727
Log. 12=1.079181 " " =—2.920819
Ans. 192=2.283300

11. What is the value of x in the following equation?

$$x=\frac{48\times.75\times72\times.0625}{.027\times120}.$$

Log. 48 = 1.681241
Log. .75 =—1.875061
Log. 72 = 1.857332
Log. .0625 =—2.795880
Log. .027=—2.431364 Arith. Com. = 1.568636
Log. 120= 2.079181 " " =—3.920819
Ans. 50=1.698969

12. What is the value of x in the following equation?

$$x=\frac{654\times320\times.3691}{87\times9\times.045}.$$ *Ans.* 2192.28.

13. What is the value of x in the following equation?

$$x=\frac{.69\times7.5\times32.71\times.003}{87\times8968\times.0008}.$$ *Ans.* .000813.

14. Multiply three hundred twenty-seven ten-thousandths by three hundred twenty-seven thousand. *Ans.* 10692.9.

15. What is the product of one thousand and twenty-five, multiplied by three hundred twenty-seven ten-thousandths? *Ans.* 33.5175.

16. Multiply .0716 by 1.326. *Ans.* .0949416.

17. Multiply .0009 by .009. *Ans.* .0000081.

INVOLUTION BY LOGARITHMS.

Rule. *Take out the logarithm of the given number from the table. Multiply the logarithm thus found by the index of the power proposed. Find the number answering to the product, and it will be the power required.*

Note.—In multiplying a logarithm with a negative index by an affirmative number, the product will be negative; but that which is to be carried from the decimal part of the logarithm will be affirmative: and, therefore, their difference will be the index of the product, and is always to be made of the same kind with the greater.

EXAMPLES.

1. What is the square of 2.579?

Logarithm of 2.579 = 0.411451
2

Ans. 6.651 = 0.822902

2. What is the third power of 32.16?

Logarithm of 32.16 = 1.507316
3

Ans. 33261.9 = 4.521948

3. Required the fourth power of .09163.

Logarithm of .09163 = —2.962038
4

Ans. .000070494 = —5.848152

Here 4 times the negative index being —8, and 3 to carry, the difference —5 is the index of the power.

EVOLUTION BY LOGARITHMS.

Rule. *Take the logarithm of the given number out of the table; divide the logarithm thus found by the index of the root; then the number answering to the quotient will be the root.*

When the index of the logarithm to be divided is negative, and does not exactly contain the divisor without some remainder, increase the index by such a number as will make it exactly divisible by the index, carrying the units borrowed, as so many *tens, to the left-hand place of the decimal, and then divide as in whole numbers.*

EXAMPLES.

1. What is the square root of 365?

Logarithm of 365 = 2.562293(2
Ans. 19.10409 = 1.281146$\frac{1}{2}$.

2. What is the third root of 12340?

Logarithm of 12340 = 4.091315(3
Ans. 23.108 = 1.363771$\frac{2}{3}$.

3. What is the seventh root of 6?

Logarithm of 6 = 0.778151(7
Ans. 1.2917 = 0.111164$\frac{3}{7}$.

4. Find the tenth root of 9.

Logarithm of 9 = 0.954243(10
Ans. 1.245 = 0.095424$\frac{3}{10}$.

5. Find the square root of .083.

Logarithm of .083 = —2.919078(2
Ans. .28809 = —1.459539.

6. Find the cube root of .00059.

Logarithm of .00059 = —4.770852(3
Ans. .083872 = —2.923617.

Here the divisor 3, not being exactly contained in —4, it is augmented by 2, to make up 6, in which the divisor is contained just 2 times; then the 2 thus borrowed, being carried to the decimal figure 7, makes 27; which, being divided by 3, gives 9, &c.

7. What is the value of x in the following equation?

$$x=\left(\frac{27\times38\times15.61}{.36\times1.37}\right)^{\frac{3}{4}}.$$

Log. 27	=1.431364
Log. 38	=1.579784
Log. 15.61	=1.193403
Log. .36=−1.556303 Arith. Com.	=0.443697
Log. 1.37= 0.136721 " "	=−1.863279
	4.511527
	3
	13.534581(4
Ans. 2419.05	=3.383645

8. Find the value of x in the following equation.

$$x=\frac{37}{223}\cdot\left(\frac{14.21\times.00208}{.035}\right)^{\frac{4}{3}}.\quad \textit{Ans.}\ .132438.$$

9. What is the value of x in the following equation?

$$x=\frac{7}{11}\left(\frac{144}{237}\right)^{\frac{2}{3}}\cdot\left(\frac{703}{819}\right)^{\frac{3}{5}}.\quad \textit{Ans.}\ .416532.$$

10. Find the value of x in the following equation.

$$x=\frac{345}{417}\cdot\left(\frac{872\times.0065}{.038\times4685}\right)^{\frac{3}{5}}.\quad \textit{Ans.}\ .10457.$$

11. What is the value of x in the following equation?

$$x=\frac{25}{476}\cdot\left(\frac{873}{956}\right)^{3}\cdot\left(\frac{278}{1973}\right)^{\frac{3}{4}}.\quad \textit{Ans.}\ .0091979.$$

12. What is the value of x in the following equation?

$$x=\frac{17}{112}\left(\frac{13.73\times.0706}{.253}\right)^{\frac{3}{2}}.\quad \textit{Ans.}\ 1.13835.$$

13. Find the value of x in the following equation.

$$x=\left(\frac{38.47\times.463}{.037\times576}\right)^{\frac{2}{3}}.\quad \textit{Ans.}\ .887264.$$

14. Required the value of x in the following equation.

$$x=\left(\frac{475\times329\times1728}{128}\right)^{\frac{1}{3}}.\quad \textit{Ans.}\ 128.2.$$

SECTION XXX.

COMPOUND INTEREST.

Art. 312. Compound Interest is interest charged not only on the principal, but also on the interest of preceding years.

Let p = principal.

r = rate per cent., considered as a decimal, or hundredths.

t = time in years.

A = amount.

Then $1+r$ will represent the amount of \$1, or 1£, for one year.

And $p\ (1+r)$ will be the amount of any principal (p) for 1 year.

The amount for two years will be $p\ (1+r)\ .\ (1+r) = p(1+r)^2$; the amount for 3 years will be $p(1+r)^2\ .\ (1+r) = p(1+r)^3$; for 4 years it will be $p(1+r)^3\ .\ (1+r) = p(1+r)^4$.

Hence, for any number of years, it will be $p(1+r)^n$; or $p(1+r)^t$.

Putting A for amount, we have the following formula for ascertaining the amount of any principal at any rate per cent. for any definite time, at compound interest.

$$A = p(1+r)^t.$$

This equation contains four quantities, A, p, r, and t; any three of which being given, the other may be obtained.

Thus, we have the following

FORMULÆ.

(1.) $A = p(1+r)^t$.

(2.) $p = \dfrac{A}{(1+r)^t}$.

(3.) $r = \left(\dfrac{A}{p}\right)^{\frac{1}{t}} - 1$.

(4.) $t = \dfrac{\log.\left(\dfrac{A}{p}\right)}{\log.\ (1+r)}$.

From the first formula, the pupil will perceive the following

Rule may be deduced for finding the amount of any sum at compound interest.

Rule. *Add* 1 *to the ratio, then raise this sum to a power whose exponent is equal to the time, multiply this power by the principal, and the product is the amount.*

By logarithms the operation is much facilitated, especially when the time is of much length.

EXAMPLES.

1. What is the amount of \$78.39 for 8 years. at 6 per cent. compound interest?

OPERATION BY THE FIRST FORMULA.

$$A=p(1+r)^t=78.39(1+.06)^8.$$

$$\begin{array}{lr}
\text{Log. } (1+r)=1.06 & =0.025306 \\
\text{Multiply by } t=8, & 8 \\
\cline{2-2}
(1+r)^t=(1.06)^8 & =0.202448 \\
\text{Log. } p=78.39 & =1.894261 \\
\cline{2-2}
\qquad A=\$124.94.\quad \textit{Ans.} & =2.096709
\end{array}$$

2. What is the amount of \$144 for 6 years, 9 months, at compound interest, at 5 per cent.?

$$\begin{array}{lr}
\text{Log. } (1+r)=1.05 & =0.021189 \\
\text{Multiply by } t, & 6 \\
\cline{2-2}
(1+r)^t=(1.05)^t & =0.127134 \\
\text{Log. } p=144 & =2.158362 \\
\cline{2-2}
\text{Log. of amount for 6 years} & =2.285496 \\
\text{Log. } (1.0375) & =0.015988 \\
\cline{2-2}
\qquad A=\$200.21.\quad \textit{Ans.} & =2.301484
\end{array}$$

We have just found the logarithm of the amount for 6 years, and to this we have added the logarithm of 1.0375, it being the amount of \$1 for 9 months, at 5 per cent.

3. What is the amount of \$500 for 9 years, at 6 per cent. per annum, the interest to be paid semi-annually?

As the time, t, is to be calculated in half-years, and as r is considered the interest of \$1 for one year, therefore $2t$ will represent the time, and $\frac{r}{2}$ the interest of \$1 for half a year. The formula will therefore be

$$A=p\left(1+\frac{r}{2}\right)^{2t}=500(1+.03)^{18}.$$

$$\begin{array}{lr}
\text{Log. } \left(1+\frac{r}{2}\right)=1.03 & =0.012837 \\
\text{Multiply by 18 half-years.} & 18 \\
\hline
\text{Log. } \left(1+\frac{r}{2}\right)^{2t} & =0.231066 \\
\text{Log. } p=500 & =2.698970 \\
\hline
\quad A=\$851.21. \quad \textit{Ans.} & =2.930036
\end{array}$$

4. What principal, at compound interest, will amount to \$4000 in 10 years, at 6 per cent.?

This question must be performed by the second formula.

$$p=\frac{A}{(1+r)^t}=\frac{4000}{(1.06)^{10}}.$$

$$\begin{array}{lr}
\text{Log. } 1.06=0.025306 & \\
\qquad\qquad\qquad\quad 10 & \\
\hline
\qquad\quad 0.253060 \text{ Arith. Com.} & =-1.746940 \\
\text{Log. } A=4000 & =3.602060 \\
\hline
\quad p=\$2233.57. \quad \textit{Ans.} & =3.349000
\end{array}$$

5. At what rate per cent. must \$2233.57 be, at compound interest, to amount to \$4000 in 10 years?

This question should be performed by the third formula.

$$r=\left(\frac{A}{p}\right)^{\frac{1}{t}}-1=\left(\frac{4000}{2233.57}\right)^{\frac{1}{10}}-1.$$

Log. $A=4000$ $= 3.602060$
Log. $p=2233.57$ $= 3.349000$

$0.253060(10$

Log. $(1+r)=1.06$ $= 0.025306$
1

.06, that is, 6 per cent. *Ans.*

6. In what time will \$2233.57, at compound interest, at 6 per cent., amount to \$4000?

This question is solved by the fourth formula.

$$t=\frac{\text{Log.}\left(\frac{A}{p}\right)}{\text{Log.}\,(1+r)}=\frac{\text{Log.}\left(\frac{4000}{2233.57}\right)}{\text{Log.}\,(1+.06)}=\frac{\text{Log. }4000-\text{log. }2233.57}{\text{Log.}\,(1+.06)}$$

Log. $A=4000$ $= 3.602060$
Log. $p=2233.57$ $= 3.349000$

0.253060

Log. $(1+r)=1.06$ 0.025306

Therefore $t=\frac{253060}{25306}=10$ years. *Ans.*

The value of this fraction can be ascertained by logarithms.
Thus, Log. 253060 $= 5.403223$
Log. 25306 $= 4.403223$

1.000000

$t=$ 10 years, as before.

7. What will \$16 amount to in 30 years, at 5 per cent. compound interest? *Ans.* \$69.15.

8. What will \$2000, at compound interest, amount to in 11 years, at 8 per cent.? *Ans.* \$4663.31.

9. What will \$27.18 amount to in 8 years, 3 months, at 4 per cent. compound interest? *Ans.* \$37.56.

10. What is the compound interest of $1728 for 8 years, 6 months, at 6 per cent. per annum, the interest to be paid every 3 months? *Ans.* $1138.74.

11. What is the amount of $18.29 for 8 years, 8 months, 12 days, at 4 per cent.? *Ans.* $25.73.

12. What sum, at compound interest, will amount to $800 in 7 years, at 5 per cent. compound interest? *Ans.* $568.54.

13. What sum will amount to $500 in 9 years, at 6 per cent. per annum, the interest to be paid every 3 months?

Ans. $292.54.5.

14. At what rate per cent. will $800, at compound interest, amount to $1609.76 in 12 years? *Ans.* 6 per cent.

15. In how many years will $3726 amount to $5007.43, at 3 per cent. compound interest? *Ans.* 10 years.

16. How many years will it require for any sum to double itself, at 6 per cent. compound interest?

Let $2p=$ the amount.

Then, $2p=p(1+r)^t$.

And $2=(1+r)^t$.

$$t=\frac{\text{Log. }2}{\text{Log. }(1+r)}.$$

Log. 2 $=0.301030$

Log. 1.06 $=0.025306$

Therefore $\frac{301030}{25306}=11.89$ years. *Ans.*

17. How many years will it require any sum to triple itself, at 5 per cent. compound interest? *Ans.* 22 years, 188 days.

18. In 1840, the number of inhabitants in the United States was 17,068,666; in 1850, the number was 23,267,498. What was the gain per cent. per annum? *Ans.* .03146 per cent.

19. At the same rate as in the last question, in what year will there be 100,000,000 inhabitants? *Ans.* May 3d, 1897.

Note. — This answer is on the presumption that the census is taken the first day of May.

20. Required the compound interest upon $155, for 9 years, at 8½ per cent. *Ans.* 56.24+.

21. Required the amount of $820 for 2½ years, at 4½ per cent. per annum, the interest being paid half-yearly. *Ans.* $916.49+.

22. What sum at compound interest, for 2½ years, at 4½ per cent., the interest payable every six months, will amount to $458.25? *Ans.* $410.02.

23. At what rate per cent. will $2000, at compound interest, amount to $4663.31 in 11 years? *Ans.* 8 per cent.

DISCOUNT AND PRESENT VALUE AT COMPOUND INTEREST.

313. Let $p =$ the present value.

$s =$ the sum due.

$t =$ the time.

$d =$ the discount.

Then, by principles before explained, we have the following

FORMULÆ.

$$(1.)\quad p=\frac{S}{(1+r)^t} \qquad (2.)\quad d=s\left(1-\frac{1}{(1+r)^t}\right).$$

EXAMPLES.

1. What is the present worth of $600, due 3 years hence, at 6 per cent. compound interest? *Ans.* $503.77.

2. John Smith, Jr., owes me $312.50, which is due 2 years hence, at 4½ per cent. compound interest. What sum will now discharge the debt? *Ans.* $286.16.

3. What is the present value of $1000, due 4 years hence, at 5 per cent. compound interest? *Ans.* $822.70.

4. What is the discount on $3700, due 10 years hence, at 5 per cent. compound interest? *Ans.* $1428.51.

5. What is the present worth of $3456, due 5 years hence, at 6 per cent. compound interest? *Ans.* $2582.52.

6. What is the discount on \$1000, due four years hence, at 6 per cent. compound interest? *Ans.* \$207.91.

7. Rented a house for 5 years, at \$400 a year, the rent to be paid quarterly. What is the present worth of this rent, at 8 per cent. compound interest? *Ans.* \$1653.47.

8. Loaned a *friend* \$100 for one year, at 2 per cent. per month, compound interest; that is, the interest is to be added to the principal each month. What is the amount at the close of the year? *Ans.* \$126.82.

9. Which is the greater present value, \$400 due three years hence, at 5 per cent. compound interest, or \$500 due 4 years hence, at simple interest? *Ans.* \$500 is better by \$71.13.

10. What sum shall I put into the Savings Bank, which pays 5 per cent. compound interest, that shall in 6 years amount to \$1000? *Ans.* \$746.21.

SECTION XXXI.

DEPOSITS.

Art. 314. A deposit is a sum of money lodged in the hands of some person or corporation, for safe keeping.

1. Deposited annually in a Savings Bank, which pays 6 per cent. compound interest, \$144 for 20 years. How much money shall I have in the bank at the end of the 20th year?

Let $a =$ the sum annually deposited.
$r =$ the rate of interest.
$t =$ the time.
$A =$ the amount.

By the rule of compound interest, the sum first deposited will amount to $144(1+.06)^{20}$, or $a(1+r)^t$; for the second year,

$144(1+.06)^{19}$, or $a(1+r)^{t-1}$; for the third year, $144(1+.06)^{18}$, or $a(1+r)^{t-2}$; for the last year, $144(1+.06)^{1}$, or $a(1+r)^{1}$.

315. We have now a regular series in Geometrical Progression, where the extremes are $a(1+r)^{t}$ and $a(1+r)^{1}$, the ratio $1+r$, to find the sum of the series.

Hence, by Art. 276, we have the following formula for obtaining the amount of the deposits.

$$A=\frac{a(1+r)[(1+r)^{t}-1]}{r}.$$

OPERATION BY LOGARITHMS.

Log. $(1+r)=1.06$		$=0.025306$
Multiply by $t=20$		$=$ 20
Log. $(1+r)^{t=20}$	3.207	$=0.506120$
Subtract	1	
Log.	2.207	$=0.343802$
Log. $(1+r)=1.06$		$=0.025306$
Log. $a=144$		$=2.158362$
Log. $r=.06=-2.778151$ Arith. Com.		$=1.221849$
	Ans. $5614.60	$=3.749319$

2. A gentleman has a daughter, who is 10 years old; and he wishes to give her, as soon as her age shall be 21 years, $2000. What sum must he deposit annually in a bank, which pays 5 per cent. compound interest, to be able to accomplish it?

316. The question given above may be solved by the following formula, which is obtained from the last by transposition, &c.

$$a=\frac{Ar}{(1+r)[(1+r)^{t}-1]}=\frac{2000\times.05}{(1.05).[(1+.05)^{11}-1]}.$$

OPERATION BY LOGARITHMS.

Log. 2000	3.301030
Log. .05	-2.698970
From	2.000000

Log. 1.05=0.021189
11

1.71=0.233079
1

Log. .71 =−1.851258
Log. 1.05 = 0.021189

Take −1.872447

Ans. \$134.14. = 2.127553

3. A gentleman, when his daughter was 10 years old, deposited for her, annually, \$134.14 in a bank, which paid 5 per cent. compound interest. This sum remained until the time of her marriage; the amount then was \$2000. What was then her age?

317. The formula for the operation of the above question is obtained from the former by transposition, &c.

$$t=\frac{\text{Log.}\left(\frac{Ar}{a(1+r)}\right)+1}{\text{Log.}\ (1+r)}=\frac{\text{Log.}\left(\frac{2000\times .05}{134.14(1+.05)}\right)+1}{\text{Log.}\ (1.05)}$$

Log. 2000 =3.301030
.05 =−2.698970

From 2.000000

Log. 1.05 =0.021189
Log. 134.14 =2.127553

Take 2.148742

0.71 =−1.851258
1

1.71 =0.232996

.232996÷.021189 = 11 years, nearly.
10+11=21 years. *Ans.*

4. A certain town in the United States, at the beginning of 1840, had 1000 inhabitants. There has been an emigration to this town each successive year, on the 1st of January, of 1000 additional inhabitants. Now, supposing the population each year to gain 3 per cent., how many inhabitants would there be in this town at the end of 10 years? *Ans.* 11,807.

5. A gentleman, at the time of his marriage, deposited in a savings' bank, for the use of his wife, the sum of $150. This he continued to do for every six months until she was fifty years old. Now, if the bank pay a semi-annual dividend of 2 per cent. compound interest, and the gentleman's wife at the time of her marriage was 25 years old, what is the amount of the deposits? *Ans.* $12,939.97.

6. If a man deposits annually in a bank $47, in how long time will it amount to $400, at 6 per cent. compound interest?

Ans. 6 years, 273 days.

7. A gentleman has a son who is 15 years old, and a daughter who is 10 years old. He intends that each of them, at the age of 21, shall have $5000 in a savings' bank, which pays an annual dividend of 4½ per cent. What sum shall he deposit annually for each?

Ans. $712.48 for the son, $345.71 for the daughter.

8. Deposited annually, in a bank which pays 4 per cent. compound interest, a certain sum, which in 10 years amounted to $300. What was the annual deposit? *Ans.* $24.02,8.

9. A certain young lady deposited $10 in a savings' bank, and this she continued every three months. Now, if the bank pays 1½ per cent. compound interest at the end of each quarter, what will be the amount of her deposits in 10 years?

Ans. $550.81.

10. Now, if the lady in the last question had deposited $40 annually at the commencement of each year, and had received 6 per cent. compound interest, would her deposits at the end of 10 years have been more or less than before?

Ans. $8.02 more.

SECTION XXXII.

EXPONENTIAL OR TRANSCENDENTAL EQUATIONS.

ART. **318.** To what power must 7 be raised to amount to 2401?

Let x be the power.

Then $7^x=2401$.

The second power of 7 is found by multiplying the logarithm of 7 by 2; and the fifth power of 7 is found by multiplying the logarithm of 7 by 5, see Art. 300; therefore the xth power of 7 is found by multiplying the logarithm of 7 by x.

We have, therefore, the following equation, the logarithm of 7 being 0.845098, and the logarithm of $2401=3.380392$.

$$x\times 0.845098=3.380392.$$

Therefore, $x=\frac{3.380392}{0.845098}=4\text{th power.}$ *Ans.*

The value of x is obtained by dividing the logarithm of the numerator by the logarithm of the denominator.

The value of the logarithms may also be obtained by subtracting the logarithm of the denominator from the logarithm of the numerator, and finding the value of the remainder. Thus,

Log. 3.380392	=	0.528967
Log. 0.845098	=	−1.926907
Ans. 4th power, as before,	=	0.602060

319. If the form of the equation be $x^x=a$, the value of x may be found by the following

RULE. *First, find by trial two numbers as near the true value of* x *as possible, and substitute them for* x *separately. Then say, As the difference of the results is to the difference of the two assumed numbers, so is the difference of the true result, and either of the former, to the difference of the true number and the supposed one belonging to the result last used. Add this dif-*

ference to the supposed number, or subtract from it, according as it may be either too little or too great, and it will give the true value nearly.

EXAMPLES.

1. What is the value of x in the following equation, $x^x=100$?

Here $x \times \log. x = \log. 100 = 2$.

We find the value of x, upon trial, to be between 3 and 4.

$\text{Log. } 3 = 0.477121$

$\text{Log. } 4 = 0.602060$

$\text{Log. } 3 \times 3 = 0.477121 \times 3 \qquad = 1.431363$

$\text{Log. } 4 \times 4 = 0.602060 \times 4 \qquad = 2.408240$

Difference of results $= 0.976877$

2.000000

1.431363

Difference from the true result $= .568637$

Therefore, $.976877 : 1 :: .568637 : .582$

$3 + .582 = 3.582 = x$ nearly.

This value of x is found, on trial, to be too small, and 3.6 is found to be too great; therefore, by substituting each of these, we have

$\text{Log. } 3.582 = 0.554126$

$\text{Log. } 3.6 = 0.556303$

$\text{Log. } 3.582 \times 3.582 = 0.554126 \times 3.582 = 1.984879$

$\text{Log. } 3.6 \times 36 = 0.556303 \times 3.6 = 2.002690$

0.017811

$3.6 - 3.582 = .018$; $2.000000 - 1.984879 = 0.015121$.

Then $.017811 : .018 :: 0.015121 : .0152$.

Therefore, $.0152 + 3.582 = 3.5972$, very nearly.

2. Given $x^x=10$ to find x.

First, let $x = 2.5$.

Then $\text{log. } 2.5 = 0.397940$.

And $0.397940 \times 2.5 \qquad = .994850.$

Secondly, let $x=2.6$.

Then log. $2.6 \qquad =0.414973.$

And $0.414973 \times 2.6 \qquad =1.078929.$

$1.078929-.994850 \qquad = .084079.$

$1.-.994850=.005150$; $2.6-2.5=.1$.

Then $.084079 : .1 :: .005150 : .006$.

$2.5+.006=2.506$, nearly.

3. Required the value of x in the following equation:

$$x^x=256. \qquad \textit{Ans. } x=4.$$

4. Given $x^x=5$ to find the value of x. *Ans.* $x=2.129$.

5. Required the value of x in the following equation:

$$7^x=343. \qquad \textit{Ans. } x=3.$$

6. Find the value of x in the following equation: $x^x=3125$.
Ans. $x=5$.

320. This rule will apply to solving questions in geometrical progression, when we wish to obtain the number of terms.

EXAMPLES.

7. If the first term is 5, the last term 405, and the ratio 3, what is the number of terms?

In Art. 274, we find $L=ar^{n-1}$, and this equation, by transposition, &c., is

$$n=\frac{\text{Log.}\left(\frac{L}{a}\right)}{\text{Log. } r}+1=\frac{\text{Log. } L-\text{Log. } a}{\text{Log. } r}+1.$$

OPERATION.

Log. $405 \qquad = \quad 2.607455$

Log. $5 \qquad = \quad 0.698970$

$\qquad 1.908485$

Log. $3=0.477121$

Log. $1.908485 \qquad = \quad 0.280688$

Log. $0.477121 \qquad = -0.678628$

$4 \quad = \quad .602060$

$4+1=5$, the number of terms. *Ans.*

8. If the first term is 4, the ratio 3, and the sum of the series 484, what is the number of terms?

In Art. 278, we find

$$S=\frac{ar^n-a}{r-1}, \text{ or } \frac{a(r^n-1)}{(r-1)}.$$

Therefore, by transposition, we have

$$\frac{\text{Log. }[a+(r-1)S]-\text{Log. }a}{\text{Log. }r}=\frac{\text{Log. }[4+(3-1)484]-\text{Log. }4}{\text{Log. }3}$$

$$=[4+(3-1)484]-4=972-4.$$

$$\begin{array}{ll} \text{Log. } 972 & = 2.987666 \\ \text{Log. } 4 & = 0.602060 \\ \hline & 2.385606 \\ \text{Log. } 3=.477121 & \\ \text{Log. } 2.385606 & = 0.377598 \\ \text{Log. } .477121 & = -0.678628 \\ \hline & = 0.698970 \end{array}$$

Ans. 5, the number of terms.

9. How long must \$78.39 be at compound interest, at 6 per cent., to amount to \$124.94? *Ans.* 8 years.

10. January 1, 1840, lent my friend John Brown \$2000, at 8 per cent. compound interest, and he agreed to pay me in 5 years; but, owing to certain circumstances, he could not pay until the amount of the note was \$4663.31. When was the note paid? *Ans.* January 1, 1851.

11. How long will it require \$800, at 6 per cent. compound interest, to amount to \$1609.76? *Ans.* 12 years.

12. Loaned \$2000, at compound interest, for 11 years, and received, interest and principal, \$4663.31. At what rate per cent. was the money lent? *Ans.* 8 per cent.

13. A gentleman agreed with another to board him for a certain number of days, on the following terms: he was to pay 3 cents for the first day's board, 9 cents for the second day, 27 cents for the third day, and so on in this ratio. The amount of the gentleman's bill was \$295.23. How many days was the gentleman boarded? *Ans.* 9 days.

SECTION XXXIII.

ANNUITIES.

Art. 321. Annuity is a term used for any periodical income arising from money lent, or from tenements, land, salaries, pensions, &c., payable from time to time, but generally by annual payments.

322. Annuities are divided into those that are in Possession, and those that are in Reversion; the former meaning such as have commenced, and the latter such as will not begin till some particular event has happened, or till after some certain time has elapsed.

323. When an annuity is forborne for some years, or the payment is not made for that time, the annuity is said to be in *arrears.*

324. An annuity may also be for a certain number of years; or it may be without any limit, and then it is called a *perpetuity.*

325. The amount of an annuity, forborne for any number of years, is the sum arising from the addition of all the annuities for that number of years, together with the interest due upon each after it became due.

326. The *present worth,* or value of an annuity, is the price or sum which ought to be given for it at the present time.

EXAMPLES.

1. A man is desirous to bequeath his son a certain sum of money, which shall be deposited in an annuity office, that pays 6 per cent., that his son may receive, at the close of each year, $100 for the term of 12 years, at which time the principal and interest shall be exhausted. What is the sum bequeathed?

Let $A =$ the sum put at interest.
$a =$ the sum taken out annually.
$r =$ the rate per cent.
$t =$ the time.

327. The amount of the sum, a, taken out at the close of the first year, would be, at the end of the time, $100(1+.06)^{11}$, or $a(1+r)^{t-1}$; that taken out at the close of the second year would amount to $100\ (1+.06)^{10}$, or $a(1+r)^{t-2}$; that taken out at the end of the third year would be $100(1+.06)^{9}$, or $a(1+r)^{t-3}$; that taken out at the end of the 12th year would be only a, or $100 without interest.

Thus, we have a regular series in Geometrical Progression, where we have the extremes, a and $a(1+r)^{t-1}$, and the ratio $(1+r)$, given to find the sum of the series.

Therefore, by Art. 277, we find the sum of the series to be

$$\frac{a(1+r)^{t-1}(1+r)-a}{r}=\frac{a(1+r)^{t}-a}{r}=\frac{a[(1+r)^{t}-1]}{r}=\text{the amount}$$

of all the sums deposited. This, by the hypothesis, must be equal to $A(1+r)^{t}$.

$$\text{Therefore, } A(1+r)^{t}=\frac{a[(1+r)^{t}-1]}{r}.$$

$$\text{By division. } A=\frac{a[(1+r)^{t}-1]}{r(1+r)^{t}}=\text{sum put at interest.}$$

We, therefore, have the first of these formulæ for finding the amount of the sums drawn out annually, or at stated periods; and the last formula for ascertaining what sum must be deposited, or put at interest.

$$A=\frac{a[(1+r)^{t}-1]}{r(1+r)^{t}}=\frac{100[(1.06)^{12}-1]}{.06(1+.06)^{12}}.$$

OPERATION BY LOGARITHMS.

Log. $1+r=1.06=0.025306$

12

$2.0122 \quad =0.303672$

1

Log. $1.0122 \quad =0.005266$

Log. $100 \quad =2.000000$

From $\quad 2.005266$

$$\begin{array}{lr} \text{Log. } (1+r)^t=(1.06)^{12} & =0.303672 \\ \text{Log. } \quad r=\ .06 & =-2.778151 \\ \hline \text{Take} & -1.081823 \\ \hline \$838.38.\ \textit{Ans.} & =2.923443 \end{array}$$

2. A gentleman deposited, in an annuity office, \$2000. How much can he receive annually, if the annuity continue 15 years, at 5 per cent. compound interest?

By transposition, &c., of the last formula, we obtain the following for ascertaining the value of the annuity, *a*.

$$a=\frac{Ar(1+r)^t}{(1+r)^t-1}.\quad=\frac{2000\times.05(1.05)^{15}}{(1.05)^{15}-1}.$$

$$\begin{array}{ll} \text{Log. } 1+r=1.05= & 0.021189 \\ & \qquad 15 \\ \hline (1+r)^t=2.0789= & 0.317835 \\ \qquad\qquad 1. & \\ \hline \end{array}$$

$$\begin{array}{lr} \text{Log. } \quad 1.0789=0.032981 \text{ Arith. Com.} & =-1.967019 \\ \text{Log. } (A)=2000 & =\ \ 3.301030 \\ \text{Log. } (r)=.05 & =-2.698970 \\ \text{Log. } (1+r)^t=(1.05)^{15} & =\ \ 0.317835 \\ \hline a=\$192.68.\ \textit{Ans.} & =\ \ 2.284854 \end{array}$$

In the operation of the above question, we find it more convenient to commence with the denominator of the formula.

3. A gentleman deposited in an annuity office, which pays 5 per cent. compound interest, \$8000; in how many years will this sum be exhausted, if he draw out, annually, \$850?

328. From the equation, $A=\frac{a[(1+r)^t-1]}{r(1+r)^t}$, we obtain, by transposition, &c.,

$$t=\frac{\text{Log.}\left(\frac{a}{a-Ar}\right)}{\text{Log. }(1+r)}.=\frac{\text{Log.}\left(\frac{850}{850-(8000\times.05)}\right)}{\text{Log. }(1.05)}.$$

Log. $(A)=850$ $=2.929419$

$Ar=8000\times.05=400$

Log. $(850-400)=450$ $=2.653213$

0.276206

Log. $(1+r)=1.05$ $=0.021189$

Therefore, $\frac{276206}{21189}=13.035=13$ years, 12 days. *Ans.*

329. But the same result will be obtained by subtracting the logarithm of the denominator from the logarithm of the numerator, and finding the number corresponding with the remainder. Thus,

Log. 276206 $=5.441233$

Log. 21189 $=4.326110$

Ans. 13.035=13 years, 12 days, $=1.115123$

4. John Smith, believing he shall live 20 years, has purchased an annuity, which affords him $500 each year. What sum has he deposited in the annuity office, which pays for deposits 5 per cent. compound interest? The principal and interest are to be exhausted at the close of the 20th year. *Ans.* $6230.81.

5. If John Smith die at the end of 10 years, what sum will remain in the office? *Ans.* $3850.27.

6. Or, if the office have agreed, for his deposit, to give him, at the close of each year, $500, and if Smith should live 30 years, what will the office lose? *Ans.* $6289.

7. A gentleman bequeathed to his wife $1728, which she deposited in an office which pays 4 per cent. compound interest. How large a sum shall she receive, annually, from the office, that the annuity may continue 10 years? *Ans.* $213.09.

8. A certain Savings Bank will pay $1\frac{1}{2}$ per cent. compound interest, semi-annually. If I deposit in this bank $4000, and take from it, at the end of every six months, $500, in what time shall I have withdrawn all my money from the bank?

Ans. 4 years, 106 days.

9. What sum shall I deposit in an annuity office, that I may draw on it every 3 months for $90? The bank pays on deposits 1 per cent. each quarter of the year, and I wish to continue drawing on the bank for 10 years. *Ans.* $2954.84.

SECTION XXXIV.

INVOLUTION OF BINOMIALS.

Art. 330. A binomial or residual quantity may be raised to any power, without the trouble of continual involution, by the following

Rule. 1. *To find the terms without the coefficients.*

The index of the first, or leading quantity, begins with the index of the given power; and, in the succeeding terms, decreases continually by 1, *in every term, to the last; and in the second, or following quantity, the indices of the terms are* 0, 1, 2, 3, 4, *&c., increasing by* 1. *That is, the first term will contain only the first part of the root, with the same index as the required power. The last term of the series will contain only the second part of the given root, raised to the intended power; but all the other intermediate terms will contain the product of some powers of both members of the root, that the powers or indices of the first or leading member will always decrease by* 1, *while those of the second member will increase by* 1.

2. *To find the coefficients.*

The first coefficient is always 1, *and the second is the same as the index of the required power; to obtain the third coefficient, multiply that of the second term by the index of the leading letter in the same term, and divide the product by* 2, *and so on; that is, multiply the coefficient of the term last found by the index of the leading quantity in that term, and divide the product by the number of terms to that place, and it will give the coefficient of the term next following. In this manner all the coefficients will be obtained.*

Note 1. — The whole number of terms will be one more than the index of the given power; and, when both terms of the root are +, all the terms of the power will be +; but, if the second term be —, all the odd terms will be +, and all the even terms —, which causes the terms to be + and — alternately.

Note 2. — The sum of the two indices in each term is always the same number, that is, the index of the required power; and, reckoning from the middle of the series, both ways, or towards the right and left, the indices of the two terms are the same figures at equal distances, but mutually changed places. Also, the coefficients are the same numbers at equal distances from the middle of the series towards the right and left; so, by whatever numbers they increase to the middle, by the same, in the reverse order, they decrease to the end.

EXAMPLES.

1. Let $a+x$ be involved to the 5th power.

The terms without the coefficients, by the first rule, will be

$$a^5,\ a^4x,\ a^3x^2,\ a^2x^3,\ ax^4,\ x^5,$$

The coefficients by the second rule will be

$$1,\ 5,\ \frac{5\times4}{2},\ \frac{10\times3}{3},\ \frac{10\times2}{4},\ \frac{5\times1}{5},=$$

$$1,\ 5,\ 10,\ 10,\ 5,\ 1.$$

Therefore, the fifth power with the coefficients is

$$a^5,\ 5a^4x,\ 10a^3x^2,\ 10a^2x^3,\ 5ax^4,\ x^5.$$

2. Involve $a-x$ to the sixth power.

Ans. The terms with the coefficients will be,

$$a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6.$$

3. Required the tenth power of $a+x$.

Ans. $\left\{\begin{array}{l} a^{10}+10a^9x+45a^8x^2+120a^7x^3+210a^6x^4+252a^5x^5 \\ +210a^4x^6+120a^3x^7+45a^2x^8+10ax^9+x^{10}. \end{array}\right.$

4. Raise $x+y$ to the seventh power.

Ans. $x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$.

5. What is the ninth power of $a-b$?

Ans. $\left\{\begin{array}{l} a^9-9a^8b+36a^7b^2-84a^6b^3+126a^5b^4-126a^4b^5+84 \\ a^3b^6-36a^2b^7+9ab^8-b^9. \end{array}\right.$

The coefficients of the first twelve powers will be found in the following

TABLE.

First	power,	1, 1
Second	"	1, 2, 1
Third	"	1, 3, 3, 1
Fourth	"	1, 4, 6, 4, 1
Fifth	"	1, 5, 10, 10, 5, 1
Sixth	"	1, 6, 15, 20, 15, 6, 1
Seventh	"	1, 7, 21, 35, 35, 21, 7, 1
Eighth	"	1, 8, 28, 56, 70, 56, 28, 8, 1
Ninth	"	1, 9, 36, 84, 126, 126, 84, 36, 9, 1
Tenth	"	1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1
Eleventh	"	1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1
Twelfth	"	1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1.

By examining the preceding table, we readily perceive the law by which the coefficients are obtained.

If we wish to obtain the coefficients of the 6th power, we add together the coefficients of the 5th power, two and two.

Thus, $1+5=6$; $5+10=15$; $10+10=20$; $10+5=15$; $5+1=6$. By this process we obtain all the coefficients of the 6th power, except the first and last, which are always 1 in every power.

To obtain the coefficients of the 10th power, we add those of the 9th. Thus,

$1+9=10$; $9+36=45$; $36+84=120$; $84+126=210$; $126+126=252$; $126+84=210$; $84+36=120$; $36+9=45$; $9+1=10$.

We therefore find the coefficients to be,

1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1.

6. Raise $a+4b$ to the third power.

Let $n=4b$.

Then $a+n=a+4b$.

The third power of $a+n$, by Art. 330, $=$

$$a^3+3a^2n+3an^2+n^3.$$

Substituting $4b$ for n, we have

$$a^3+3a^2(4b)+3a(4b)^2+(4b)^3=$$
$$a^3+12a^2b+48ab^2+64b^3. \quad \textit{Ans.}$$

7. What is the third power of $a+b+c$?

Let $\quad n=b+c.$

Then $\quad a+n=a+b+c.$

The third power of $\quad a+n=a^3+3a^2n+3an^2+n^3.$

Substituting the values of n, we have

$$a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3=$$
$$\textit{Ans.} \left\{ \begin{array}{l} a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2 \\ +b^3+3b^2c+3bc^2+c^3. \end{array} \right.$$

8. What is the 3d power of $a+b+c+d$?

Let $\quad x=a+b$, and $y=c+d$.

Then $\quad (x+y)^3=(a+b+c+d)^3.$

And $\quad (x+y)^3=(x^3+3x^2y+3xy^2+y^3).$

Substituting these values of x and y, we have

$$(a+b)^3+3(a+b)^2(c+d)+3(a+b)(c+d)^2+(c+d)^3=$$
$$a^3+3a^2b+3ab^2+b^3+(3a^2+6ab+3b^2)(c+d)+(3a+3b)(c^2+2cd+d^2)+(c^3+3c^2d+3cd^2+d^3)=$$
$$a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3a^2d+6abd+3b^2d+3ac^2+6acd+3ad^2+3bc^2+6cbd+3bd^2+c^3+3c^2d+3cd^2+d^3. \quad \textit{Ans.}$$

9. What is the 3d power of $2a-b+c^2$? *Ans.*

10. What is the 5th power of $4a-5b$? *Ans.*

11. What is the 6th power of $3a^2-2b^3$? *Ans.*

12. What is the 4th power of $m+n-p$? *Ans.*

13. What is the 8th power of m^2+n^3? *Ans.*

14. What is the 7th power of $1+x^2$? *Ans.*

15. What is the 2d power of $a+b+c+d+e+f$? *Ans.*

16. What is the 10th power of a^3+b^3? *Ans*

17. What is the nth power of $a+b$? *Ans.*

18. What is the 6th power of $a-b+c$? *Ans.*

19. What is the 4th power of a^5-x? *Ans.*

20. What is the 3d power of $2a^2-3b^3$? *Ans.*

SECTION XXXV.

BINOMIAL THEOREM.

Art. 331. The Binomial Theorem is a general algebraical expression or formula, by which any power or root of a given quantity, consisting of two terms, is expanded into a series; the form of which, as it was first proposed by Sir Isaac Newton. being as follows:

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}[1+\frac{m}{n}Q+\frac{m}{n}\left(\frac{m-n}{2n}\right)Q^2+\frac{m}{n}\left(\frac{m-n}{2n}\right)\left(\frac{m-2n}{3n}\right)Q^3+\frac{m}{n}\left(\frac{m-n}{2n}\right)\left(\frac{m-2n}{3n}\right)\left(\frac{m-3n}{4n}\right)Q^4+\&c.$$

Or,

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ+\frac{m-3n}{4n}DQ+\&c.,$$

where P is the first term of a binomial, Q the second divided by the first, $\frac{m}{n}$ the index of the power or root, and A, B, C, &c., the terms immediately preceding those in which they are first found, including their signs $+$ or $-$.

332. This theorem may be applied to any particular case, by substituting the numbers or letters in the given example for P Q, m, and n, in either the above formulæ, and then finding the result according to the rule.

When the index of the binomial is a whole number, the series will terminate, as observed under the article Involution; but when it is a negative or fractional number, as in the following examples, the series will proceed on *ad infinitum*, and will become more convergent the less the second term of a binomial is with respect to the first.

EXAMPLES.

1. It is required to convert $(a^2+x)^{\frac{1}{2}}$ into an infinite series.

Let $P=a^2$, $Q=\frac{x}{a^2}$, $\frac{m}{n}=\frac{1}{2}$, or $m=1$ and $n=2$.

Then $P^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{\frac{1}{2}}=a=A$.

$$\frac{m}{n}AQ=\frac{1}{2}\times\frac{a}{1}\times\frac{x}{a^2}=\frac{x}{2a}=B.$$

$$\frac{m-n}{2n}BQ=\frac{1-2}{4}\times\frac{x}{2a}\times\frac{x}{a^2}=-\frac{x^2}{2.4a^3}=C.$$

$$\frac{m-2n}{3n}CQ=\frac{1-4}{6}\times-\frac{x^2}{2.4a^3}\times\frac{x}{a^2}=\frac{3x^3}{2.4.6a^5}=D.$$

$$\frac{m-3n}{4n}DQ=\frac{1-6}{8}\times\frac{3x^3}{2.4.6a^5}\times\frac{x}{a^2}=-\frac{3.5x^4}{2.4.6.8a^7}=E.$$

$$\frac{m-4n}{5n}EQ=\frac{1-8}{10}\times-\frac{3.5x^4}{2.4.6.8a^7}\times\frac{x}{a^2}=\frac{3.5.7x^5}{2.4.6.8.10a^9}=F.$$

Therefore $(a^2+x)^{\frac{1}{2}}=$

$$a+\frac{x}{2a}-\frac{x^2}{2.4a^3}+\frac{3x^3}{2.4.6a^5}-\frac{3.5x^4}{2.4.6.8a^7}+\frac{3.5.7x^5}{2.4.6.8.10a^9}-, \text{ \&c.}$$

The pupil will readily perceive that the law of formation of the several terms of the series is sufficiently evident.

2. Required the development of $\frac{x^2}{(x^2-y)^{\frac{1}{2}}}$ in a series.

Here $\frac{x^2}{(x^2-y)^{\frac{1}{2}}}=x^2(x^2-y)^{-\frac{1}{2}}$, $P=x^2$, $Q=-\frac{y}{x^2}$, $m=-1$, and $n=2$.

Hence $P^{\frac{m}{n}}=(x^2)^{\frac{m}{n}}=(x^2)^{-\frac{1}{2}}=\frac{1}{x}=A$.

$$\frac{m}{n}AQ=-\frac{1}{2}\times\frac{1}{x}\times-\frac{y}{x^2}=\frac{y}{2x^3}=B.$$

$$\frac{m-n}{2n}BQ=\frac{-1-2}{4}\times\frac{y}{2x^3}\times-\frac{y}{x^2}=\frac{3y^2}{2.4x^5}=C.$$

$$\frac{m-2n}{3n}CQ=\frac{-1-4}{6}\times\frac{3y^2}{2.4x^5}\times-\frac{y}{x^2}=\frac{3.5y^3}{2.4.6x^7}=D.$$

$$\frac{m-3n}{4n}DQ=\frac{-1-6}{8}\times\frac{3.5y^3}{2.4.6x^7}\times-\frac{y}{x^2}=\frac{3.5.7y^4}{2.4.6.8x^9}=E.$$

Therefore, $\frac{1}{(x^2-y)^{\frac{1}{2}}}=\frac{1}{x}+\frac{y}{2x^3}+\frac{3y^2}{2.4x^5}+\frac{3.5y^3}{2.4.6x^7}+\frac{3.5.7y^4}{2.4.6.8x^9}+$, &c.

And, $\frac{x^2}{(x^2-y)^{\frac{1}{2}}}=x+\frac{y}{2x}+\frac{3y^2}{2.4x^3}+\frac{3.5y^3}{2.4.6x^5}+\frac{3.5.7y^4}{2.4.6.8x^7}+$, &c.

This last equation is obtained from the former by multiplying each term of the equation by x^2.

3. Required the cube root of 9.

Here, $9^{\frac{1}{3}}=(8+1)^{\frac{1}{3}}$.

Therefore, $P=8$, $Q=\frac{1}{8}$, $m=1$, and $n=3$.

Whence, $P^{\frac{m}{n}}=8^{\frac{m}{n}}=8^{\frac{1}{3}}=2=A$.

$$\frac{m}{n}AQ=\frac{1}{3}\times2\times\frac{1}{2^3}=\frac{1}{3.2^2}=B.$$

$$\frac{m-n}{2n}BQ=\frac{1-3}{6}\times\frac{1}{3.2^2}\times\frac{1}{2^3}=-\frac{1}{3.6.2^4}=C.$$

$$\frac{m-2n}{3n}CQ=\frac{1-6}{9}\times-\frac{1}{3.6.2^4}\times\frac{1}{2^3}=\frac{5}{3.6.9.2^7}=D.$$

$$\frac{m-3n}{4n}DQ=\frac{1-9}{12}\times\frac{5}{3.6.9.2^7}\times\frac{1}{2^3}=-\frac{5.8}{3.6.9.12.2^{10}}=E.$$

Therefore,

$$9^{\frac{1}{3}}=2+\frac{1}{3.2^2}-\frac{1}{3.6.2^4}+\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}+\text{, \&c.}$$

4. What is the square root of $a+b$?

Here, $\frac{m}{n}=\frac{1}{2}$, $P=a$, and $Q=\frac{b}{a}$.

Then, $P^{\frac{m}{n}}=a^{\frac{1}{2}}=A$

$$\frac{m}{n}AQ=\frac{1}{2}\times\frac{a^{\frac{1}{2}}}{1}\times\frac{b}{a}=\frac{a^{\frac{1}{2}}b}{2a}=\frac{a^{-\frac{1}{2}}b}{2}=B.$$

$$\frac{m-n}{2n}BQ=\frac{1-2}{4}\times\frac{a^{-\frac{1}{2}}b}{2}\times\frac{b}{a}=-\frac{a^{-\frac{1}{2}}b^2}{8a}=-\frac{a^{-\frac{3}{2}}b^2}{8}=C.$$

$$\frac{m-2n}{3n}CQ=\frac{1-4}{6}\times-\frac{a^{-\frac{3}{2}}b^2}{8}\times\frac{b}{a}=\frac{3a^{-\frac{3}{2}}b^3}{48a}=\frac{a^{-\frac{5}{2}}b^3}{16}=D.$$

And $$\frac{m-3n}{4n}DQ=\frac{1-6}{8}\times\frac{a^{-\frac{5}{2}}b^3}{16}\times\frac{b}{a}=\frac{5a^{-\frac{5}{2}}b^4}{128a}=-\frac{5a^{-\frac{7}{2}}b^4}{128}=E.$$

Therefore, $(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{a^{-\frac{1}{2}}b}{2}-\frac{a^{\frac{3}{2}}b^2}{8}+\frac{a^{\frac{5}{2}}b^3}{16}-\frac{5a^{-\frac{7}{2}}b^4}{128}$, &c.

5. What is the cube root of 7?

Ans. $2-\frac{1}{3.2^2}-\frac{1}{3.6.2^4}-\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}-$, &c.

6. Expand $(1-a)^{\frac{2}{5}}$ into an infinite series.

Ans. $1-\frac{2a}{5}-\frac{2.3.a^2}{5.10}-\frac{2.3.8.a^3}{5.10.15}-$, &c.

7. It is required to convert $\frac{1}{(1+x)^{\frac{1}{5}}}$, or its equal $(1+x)^{-\frac{1}{5}}$, into an infinite series.

Ans. $1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11x^3}{5.10.15}+\frac{6.11.16x^4}{5.10.15.20}-$, &c.

8. It is required to convert $(a-b)^{\frac{1}{4}}$ into an infinite series.

Ans. $a^{\frac{1}{4}}\left(1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7.b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-\text{, \&c.}\right)$

INDETERMINATE COEFFICIENTS.

333. This is a general method of obtaining a series from fractions, and other expressions, without either performing the division or extracting the root.

RULE. *Assume a series with unknown but constant coefficients of* x, *increasing or decreasing in the same way as if the operation was performed at length; then make this series equal to the given expression, and, clearing the equation of* *fractions, bring all the terms to one side, so as to make the equation* $= 0$; *next make the first term of the coefficients of the several powers of* x *each* $= 0$, *and there will arise as many independent equations as there are unknown coefficients, from which their values may be found and substituted for them in the assumed series.*

EXAMPLES.

1. Let it be required to expand $\frac{a}{b+x}$ into a series.

Assume $\frac{a}{b+x}=A+Bx+Cx^2+Dx^3+$&c.; then, multiplying both sides by $b+x$, and transposing a, we obtain $Ab-a+(Bb+A)x+(Cb+B)x^2+(Db+C)x^3+$&c.$=0$, an equation which must be true, whatever be the value of x. Now, making the first term, and the coefficients of the several powers of x, each $= 0$, we have $Ab-a=0$, or $A=\frac{a}{b}$; $Bb+a=0$, or $B=\frac{A}{b}=-\frac{a}{b^2}$; $Cb+B=0$, or $C=\frac{B}{b}=+\frac{a}{b^3}$; $Db+c=0$, or $D=\frac{C}{b}=-\frac{a}{b^4}$, &c. And, substituting these values of A, B, C, D, &c., in the assumed series, we get $\frac{a}{b+x}=\frac{a}{b}-\frac{ax}{b^2}+\frac{ax^2}{b^3}-\frac{ax^3}{b^4}+$ &c., in which, it is obvious, that the signs are alternately $+$ and $-$, and the exponents, both in the numerator and denominator, increase continually by 1, that of x in the numerator being always 1 less than that of b in the denominator.

2. Expand $\frac{a^2}{a^2+2ax-x^2}$ into a series.

Ans. $1-\frac{2x}{a}+\frac{5x^2}{a^2}-\frac{12x^3}{a^3}+$, &c.

3. Expand $\sqrt{(a^2-x^2)}$ into a series.

Ans. $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-$, &c.

4. Expand $\frac{1+2x}{1-x-x^2}$ into a series.

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5+$, &c.

This is a recurring series, in which each of the coefficients, after the second, is the sum of the two preceding ones.

5. Expand $\sqrt{(1-a)}$ into a series.

Ans. $1-\frac{a}{2}-\frac{a^2}{2.4}-\frac{3a^3}{2.4.6}-\frac{3.5a^4}{2.4.6.8}-\frac{3.5.7a^5}{2.4.6.8.10}-$, &c.

6. Expand $\frac{1-x}{1-2x-3x^2}$ into a series.

Ans. $1+x+5x^2+13x^3+41x^4+121x^5+365x^6$, &c.

7. What is the expansion of $(a-b)^{\frac{1}{4}}$?

Ans. $a^{\frac{1}{4}}\left(1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-\right.$, &c.$\left.\right)$

8. It is required to expand $(a+x)^{-2}$.

Ans. $\frac{1}{a^2}-\frac{2x}{a^3}+\frac{3x^2}{a^4}-\frac{4x^3}{a^5}+$, &c.

9. It is required to expand $\frac{1}{(a+2x)^3}$.

Ans. $\frac{1}{a^3}-\frac{6x}{a^4}+\frac{24x^2}{a^5}-\frac{80x^3}{a^6}$, &c.

10. It is required to find the expansion of $\frac{2}{(c+x)^2}$.

Ans. $\frac{2}{c^2}-\frac{4x}{c^3}+\frac{6x^2}{c^4}-\frac{8x^3}{c^5}+$, &c.

11. It is required to find the expansion of $\frac{a^2}{(a+2b)^3}$.

Ans. $\frac{1}{a}\left(1-\frac{6b}{a}+\frac{24b^2}{a^2}-\frac{80b^3}{a^3}+\right.$, &c.$\left.\right)$

12. What is the value of $\frac{1}{(b^2+x)^{\frac{1}{2}}}$ in a series?

Ans. $\frac{1}{b}-\frac{x}{2b^3}+\frac{3x^2}{2.4b^5}-\frac{3.5x^3}{2.4.6b^7}+\frac{3.5.7x^4}{2.4.6.8b^9}-$, &c.

SECTION XXXVI.

SUMMATION AND INTERPOLATION OF SERIES.

Art. 334. The Summation of Series is the method of finding a terminated expression equal to the whole series.

Interpolation is the method of finding any term of an infinite series, without producing all the rest.

DIFFERENTIAL METHOD.

335. The Differential Method consists in finding, from the successive differences of the terms of a series, any intermediate term, or the sum of the whole series.

Problem I.

336. To find the several orders of differences.

Let $a+b+c+d+e+$, &c., be any series; subtract each term from the one following it, and the differences $-a+b$, $-b+c$, $-c+d$, $-d+e$, &c., will form a new series, called the *first order of differences.* Again, subtract each term of this new series from the one that follows it, and the differences $a-2b+c$, $b-2c+d$, $c-2d+e$, &c., will form another series, called the *second order of differences.* Proceed in like manner for the third, fourth, fifth, &c., order of differences, until they at last become equal to 0, or are carried as far as is required.

337. When the several terms of the series continually increase, the differences will all be positive; but, when they decrease, the differences will be alternately negative and positive.

1. Required the several order of differences of the series 1, 6, 20, 50, 105, 196, &c.

1, 6, 20, 50, 105, 196, &c., the given series.
5, 14, 30, 55, 91, &c., 1st differences.
9, 16, 25, 36, &c., 2d "
7, 9, 11, &c., 3d "
2, 2, &c., 4th "
0, &c., 5th "

2. Required the several order of differences of the series of 1^2, 2^2, 3^2, 4^2, 5^2, &c.

1, 4, 9, 16, 25, &c., the given series.
3, 5, 7, 9, &c., 1st differences.
2, 2, 2, &c., 2d "
0, 0, &c., 3d "

3. Required the several order of differences of the series of cubes, 1^3, 2^3, 3^3, 4^3, 5^3. *Ans.*

4. Find the order of differences in the series $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c. *Ans.*

PROBLEM II.

338. To find the first term of any order of differences.

Let d', d'', d''', d'''', &c., represent the first terms of the 1st, 2d, 3d, 4th, &c., order of differences; then $d'=-a+b$, $d''=a-2b+c$, $d'''=-a+3b-3c+d$, $d''''=a-4b+6c-4d+e$, &c.; from which it is obvious that the coefficients of the several terms of any order of differences are respectively the same as those of the terms of an expanded binomial, and are obtained in the same manner; for the terms that are subtracted are actually added, but with contrary signs. Hence we infer that d^n, or the first difference of the nth order of differences, is $\pm a\mp nb\pm n.\frac{n-1}{2}c\mp n.\frac{n-1}{2}.\frac{n-2}{3}d\pm$, &c., to $n+1$ terms; in which formula the *upper* signs must be taken when n is an even number, and the *under* when n is an odd number.

5. Required the first of the fifth order of differences of the series 6, 9, 17, 35, 63, 99, 148, &c.

Let a, b, c, d, e, f, &c. $=$ 6, 9, 17, 35, 63, 99, 148, &c., and $n=5$. Then

$$-a+nb-\frac{n(n-1)}{2}c+\frac{n(n-1)(n-2)}{2.3}d-\frac{n(n-1)(n-2)(n-3)}{2.3.4}e$$
$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{2.3.4.5}f=-a+5b-\frac{5.4}{2}c+\frac{5.4.3}{2.3}d-$$

$\frac{5.4.3.2}{2.3.4}e+\frac{5.4.3.2.1}{2.3.4.5}f=-6+45-170+350-315+99=$ $494-491=+3$. *Ans.*

6. Required the first of the sixth order of differences of the series 3, 6, 11, 17, 24, 36, 50, 72, &c. *Ans.* −14.

PROBLEM III.

339. To find the nth term of the series a, b, c, d, e, f, &c.

As we have found in the last problem that $d'=-a+b$, therefore $b=a+d'$, and, in the same manner, we find $c=a+2d'+d''$, $d=a+3d'+3d''+d'''$, $e=a+4d'+6d''+4d'''+d''''$, &c.; whence the nth term is

$$=a+\frac{n-1}{1}d'+\frac{n-1}{1}.\frac{n-2}{2}d''+\frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}d'''+, \&c.$$

7. Required the 7th term of the series 3, 5, 8, 12, 17, &c.

3, 5, 8, 12, 17, &c., the given series.
2, 3, 4, 5, 1st difference.
1, 1, 1, 2d difference.
0, 0, 3d difference.

Here $d'=2$, $d''=1$, $d'''=0$, and $n=7$.

Therefore $a+\frac{n-1}{1}d'+\frac{n-1}{1}.\frac{n-2}{2}d''=3+\frac{7-1}{1}.2+\frac{7-1}{1}.\frac{7-2}{2}.1=3+12+15=30=$ the 7th term.

8. Required the 9th term of the series 1, 5, 15, 35, 70, &c. *Ans.* 495.

9. Required the 10th term of the series 1, 3, 6, 10, 15, 21, &c. *Ans.* 55.

PROBLEM IV.

340. To find the sum of n terms of the series a, b, c, d, e, &c.

If we add the values of a, b, c, &c., as found in the last problem, we obtain $2a+d'=a+b$, $3a+3d'+d''=a+b+c$, $4a+$

$6d'+4d''+d'''=a+b+c+d$, &c. Wherefore it is evident that the sum of n terms must be

$$na+n.\frac{n-1}{2}d'+n.\frac{n-1}{2}.\frac{n-2}{3}d''+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}d'''+,$$

&c.

341. When the differences become at last $=0$, any term, or the sum of any numbers, can be accurately found; but, when the differences do not vanish, the formulæ in this and the preceding problem give only an approximation, which will come nearer the truth as the differences diminish.

10. Required the sum of 8 terms of the series 2, 5, 10, 17, &c.

Here $n=8$, $a=2$, $d'=3$, $d''=2$, and $d'''=0$.

Hence, $na+n.\frac{n-1}{2}d'+n.\frac{n-1}{2}.\frac{n-2}{3}d''=8.2+8.\frac{7}{2}.3+8.\frac{7}{2}.\frac{6}{3}.2=16+84+112=212=$ the sum of 8 terms.

11. Required the sum of 100 terms of the series 1, 2, 3, 4, 5, &c.

Here 1, 2, 3, 4, 5, 6, &c., given series.
1, 1, 1, 1, 1, &c., 1st difference.
0, 0, 0, 0, &c., 2d difference.

Here $n=100$, $a=1$, and $d=1$.

$$na+n.\frac{n-1}{2}d=100+100.\left(\frac{100-1}{2}\right)1=5050. \quad \textit{Ans.}$$

12. Required the sum of 12 terms of the series, 1, 4, 10, 20, 35. *Ans.* 1365.

13. Required the sum of n terms of the series 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, &c.

Here 1, 4, 9, 16, 25, 36, 49, &c., given series.
3, 5, 7, 9, 11, 13, &c., 1st difference.
2, 2, 2, 2, 2, &c., 2d difference.
0, 0, 0, 0, &c., 3d difference.

Let $a=1$, $d'=3$, and $d''=2$.

Then $na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2\,.\,3}d''=\frac{n(n+1)\,.\,(2n+1)}{6}$.

14. Required the sum of n times of the series

$1^3, 2^3, 3^3, 4^3, 5^3, 6^3$, &c.; 1, 8, 27, 64, 125, 216, &c.

Here 1, 8, 27, 64, 125, 216, &c., given series.

7, 19, 37, 61, 91, &c., 1st difference.

12, 18, 24, 30, &c., 2d difference.

6, 6, 6, &c., 3d difference.

0, 0, &c., 4th difference.

Let $a=1$, $d'=7$, $d''=12$, $d'''=6$. Then

$$na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2\,.\,3}d''+\frac{n(n-1)(n-2)(n-3)}{2\,.\,3\,.\,4}d.'''$$

$$=n+\frac{7n(n-1)}{2}+\frac{12n(n-1)(n-2)}{2\,.\,3}+\frac{6n(n-1)(n-2)(n-3)}{2\,.\,3\,.\,4}$$

$$=n+\frac{7n^2-7n}{2}+2n^3-6n^2+4n+\frac{n^4-6n^3+11n^2-6n}{4}=$$

$$\frac{4n}{4}+\frac{14n^2-14n}{4}+\frac{8n^3-24n^2+16n}{4}+\frac{n^4-6n^3+11n^2-6n}{4}=$$

$$\frac{n^4+2n^3+n^2}{4}=\frac{n^2(n+1)^2}{4}=$$ sum of n terms, as required.

15. What is the number of cannon-shot in a square pile, the bottom row consisting of 25 shot*? *Ans.* 5525.

16. I have 10 square house-lots, whose sides measure 5, 6, 7, 8, 9, &c., rods, respectively. What is their value, at 25 cents per square foot? *Ans.* \$67,041,56¼.

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* Shots and shells are generally piled in three different forms, called triangular, square, or oblong piles, according as their base is either a triangle, a square, or a rectangle.

A square pile is formed by the continual laying of square, horizontal courses of shot, one above another, in such a manner as that the sides of their courses decrease by unity from the bottom to the top row, which ends also in one shot.

17. There are 5 cubical blocks of marble, whose sides measure, respectively, 2, 3, 4, 5, and 6 feet? What is their value at $2.75 per cubic foot? *Ans.* $1210.

18. What is the number of shot in a square pyramidical pile, whose side at the base contains 100 shot? *Ans.* 338350.

19. What is the sum of 20 terms of the series 1^3, 2^3, 3^3, 4^3, 5^3, 6^3, &c.? *Ans.* 44100.

20. What is the sum of 20 terms of the series 1^4, 2^4, 3^4, 4^4, 5^4, 6^4, &c.? *Ans.* 722666.

PROBLEM V.

342. To find a fraction that will express the value of a geometrical series to infinity.

In Art. 284 we find that the sum of an infinite series is obtained by the following formula:

$$S=\frac{a}{1-r};$$

and, by this formula, we may find the sum of algebraic series.

EXAMPLES.

1. What is the sum of the series $1+a+a^2+a^3+a^4$, &c., carried to infinity? *Ans.* $\frac{1}{1-a}$.

By the above formula, the first term of the series will be the numerator of the fraction, and the denominator is obtained by subtracting the second term from the first.

2. What fraction will express the exact value of the series $1+5+25+125$, &c., to infinity? *Ans.* $\frac{1}{1-5}$.

3. What fraction will express the infinite series $1-a+a^2-a^3+a^4-a^5$, &c.? *Ans.* $\frac{1}{1+a}$.

4. What fraction will express the series $\frac{h}{a}+\frac{bh}{a^2}+\frac{b^2h}{a^3}+$, &c., to infinity? *Ans.* $\frac{h}{a-b}$.

5. What is the sum of the series $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+$, &c., to infinity? *Ans.* $\frac{1}{x-1}$.

6. What fraction will express the series $1+2+4+8+16$, &c., to infinity? *Ans.* $\frac{1}{1-2}$.

7. What fraction is equal to the series $\frac{1}{a}-\frac{x^2}{a^3}+\frac{x^4}{a^5}-\frac{x^6}{a^7}+$, &c., to infinity? *Ans.* $\frac{a}{a^2+x^2}$.

8. What fraction will express the value of $1+1+1+1$, &c., to infinity? *Ans.* $\frac{1}{1-1}$.

9. Express by a fraction the value of the series $x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+$, &c., to infinity. *Ans.* $\frac{ax}{a-x}$.

10. What is the value of the series $1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+$, &c., to infinity? *Ans.* $\frac{a}{a+x}$.

11. Required the sum of the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+$, &c., continued to infinity. *Ans.* 1.

This question may be performed by separating the factors of the denominators so as to form two series, and then subtracting the less from the greater, as follows:

Let $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, &c. = the greater series.
And $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, &c. = the less series.
Then $1 =$ the sum of the series.

NOTE. — Another method may be found in the Key.

12. Required the sum of the series $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+$, &c., to infinity. *Ans.* $\frac{11}{18}$.

SECTION XXXVII.

CUBIC EQUATIONS, CONTAINING ONLY THE THIRD AND SECOND POWERS.

Art. 343. Any numerical equation, containing only the third and second powers of the unknown quantity, and having one rational root, may be reduced by rendering both of its members perfect squares, and extracting the square root of both sides; completing the operation by former rules. The only difficulty lies in multiplying the equation by such a number that, after adding to each side the fourth power of the unknown quantity, and the second power with a coefficient easily determined, both sides will be perfect squares. This multiplier must be ascertained by trial; for, though a general formula might be given for obtaining it, yet it would be so complicated as to be of no practical use. It may be either an integer or a fraction, and is positive or negative according to the sign of the known quantity.

Though there always is such a multiplier whenever the unknown quantity has one rational value, yet, when the numbers are very large, or the equation is very complicated, it may not be readily found, and the process of trial may become too tedious to be of service. Whenever the equation does not contain too large numbers, the pupil will find little difficulty, if he thoroughly understands the following

Rule. *Divide both sides of the equation by the coefficient of the unknown cube, if it have any expressed. Place the third power of the unknown quantity on one side of the equation, and the second power, with the known quantity, on the other. Multiply both sides by the number nearest to unity which will make the known quantity a positive square; or, which is the same thing, separate the known number into two factors, one of which shall be the greatest square contained in it, and multiply both sides by the other factor.*

Multiply the last equation by 4; *add the fourth power of the* *unknown quantity, and the second power, with a coefficient equal to the square of half the coefficient of the third power, to each side; and extract the square root of both sides, if possible. By taking like signs of the two members of the equation in evolving, we shall obtain one root; and, by taking unlike signs, the other two may be found by quadratic equations.*

But, if that member of the equation which contains the known quantity is not a perfect square, substitute 1, 9, 16, $\frac{9}{4}$, $\frac{4}{9}$, $\frac{1}{4}$, $\frac{1}{9}$, *or some other square number, in the place of* 4, *and proceed as above, till, by trial, a number is found which will accomplish the object.*

NOTE.—1. The sum of the three values of the unknown quantity should always be equal to the coefficient of the second power in the original equation, after dividing by the coefficient of the cube, and placing it on the same side with the known quantity, opposite the positive cube; hence, if two values were known, the other might easily be found.

2. When one of the values is known, the others might be found by the usual method; bringing all the terms of the original equation to the same side, and dividing by the difference between the unknown quantity and its known value, reducing, by quadratic equations, the equation thus produced. But the three values are here given directly, by using the different signs in evolving, thus rendering the solution shorter, and more satisfactory. It is evident that, in extracting the square root of an equation, both sides may be considered positive, or both negative, or either one positive and the other negative. Thus the square root of the equation $4a^2-8ab+4b^2=c^2+2cd+d^2$, is $+(2a-2b)=+(c+d)$, or $-(2a-2b)=-(c+d)$, or $+(2a-2b)=-(c+d)$, or $-(2a-2b)=+(c+d)$. But, if both sides take like signs, the result will be the same, whether they are both positive or both negative, as the signs of both sides of an equation may always be changed; while, if they take unlike signs, a different equation will be produced, it making no difference which side is positive. Hence, there are but two results that can be obtained, and we have preferred to express them, in the following examples, by the same method as in quadratic equations; prefixing the sign ± to the right-hand member of the equation produced by evolution.

3. By observing whether the root of the known quantity is greater or less than half the coefficient of the second power on the same side, if we also notice the sign, we may usually know whether the multiplier we have used is too small or too large. When there are two rational values of the unknown quantity, of course the third will be rational, and there will be three different multipliers, which will answer our purpose, thus giving three different solutions for the same example.

EXAMPLES.

1. What are the values of x in the equation $x^3-x^2=4$?

Here the multiplier, which would make the known quantity a perfect square, is unity; therefore we transpose, and multiply by 4, $4x^3=4x^2+16$.

Adding x^4 and $\left(\frac{4}{2}\right)^2x^2$ to each side, $x^4+4x^3+4x^2=x^4+8x^2+16$.

Evolving, $x^2+2x=\pm(x^2+4.)$

Taking the positive sign and cancelling, $2x=4$.

Dividing, $x=2$.

Taking the negative sign, $x^2+2x=-x^2-4$.

Transposing and dividing, $x^2+x=-2$.

By quadratics, $x=\frac{-1\pm\sqrt{-7}}{2}$.

Hence $x=2$, or $\frac{-1\pm\sqrt{-7}}{2}$. *Ans.*

The sum of their values, $2+\frac{-1+\sqrt{-7}}{2}+\frac{-1-\sqrt{-7}}{2}$, is 1.

2. What are the values of x in the equation $4x^3+10x^2=9$?

Conditions, $4x^3+10x^2=9$.

Dividing by 4 and transposing, $x^3=-\frac{5}{2}x^2+\frac{9}{4}$.

$\frac{9}{4}$ being a square, multiply by 4, $4x^3=-10x^2+9$.

Adding x^4 and $\left(\frac{4}{2}\right)^2x^2$, $x^4+4x^3+4x^2=x^4-6x^2+9$.

Evolving, $x^2+2x=\pm(x^2-3)$.

Taking the positive sign and cancelling, $2x=-3$.

Dividing, $x=-\frac{3}{2}$.

Taking the negative sign, $x^2+2x=-x^2+3$.

Transposing and dividing, $x^2+x=\frac{3}{2}$.

By quadratics, $x=\frac{-1\pm\sqrt{7}}{2}$.

Hence $x=-\frac{3}{2}$, or $\frac{-1\pm\sqrt{7}}{2}$. *Ans.*

The sum of these roots is $-\frac{5}{2}$.

3. Given $3x^3-2x^2=931$ to find the values of x.

(1.) Conditions, $3x^3-2x^2=931$.

(2.) Dividing and transposing, $x^3=\frac{2x^2}{3}+\frac{931}{3}$.

(3.) The greatest square in $\frac{931}{3}$ is $\frac{49}{1}$;

therefore $\frac{931}{3}=\frac{49}{1}\times\frac{19}{3}$.

Multiplying (2) by $\frac{19}{3}$, $\frac{19x^3}{3}=\frac{38x^2}{3}+\frac{17689}{9}$.

Multiplying by 4, $\frac{76x^3}{3}=\frac{152x^2}{3}+\frac{70756}{9}$.

Adding x^4 and $\left(\frac{38}{3}\right)^2x^2$, $x^4+\frac{76x^3}{3}+\frac{1444x^2}{9}=x^4+\frac{532x^2}{3}+\frac{70756}{9}$.

Evolving, $x^2+\frac{38x}{3}=\pm\left(x^2+\frac{266}{3}\right)$.

Taking the positive sign and cancelling, $38x=266$.

Dividing, $x=7$.

Taking the negative sign, $x^2+\frac{38x}{3}=-x^2-\frac{266}{3}$.

Transposing and dividing, $x^2+\frac{19x}{3}=-\frac{133}{3}$.

By quadratics, $x=\frac{-19\pm\sqrt{-1235}}{6}$.

Hence $x=7$, or $\frac{-19\pm\sqrt{-1235}}{6}$. The sum of these is $\frac{2}{3}$.

4. Given $x^3=12x^2-81$ to find the values of x.

Conditions, $x^3=12x^2-81$.

By multiplying both sides by -1, -81 becomes a positive square, $-x^3=-12x^2+81$.

We find that neither 4, 9, 16, nor 25, will answer our purpose, and we multiply by 36, $-36x^3=-432x^2+2916$.

Adding x^4 and $\left(\frac{36}{2}\right)^2x^2$, $x^4-36x^3+324x^2=x^4-108x^2+2916$.

Evolving, $x^2-18x=\pm(x^2-54)$.

Taking the positive sign and cancelling, $-18x=-54$.

Changing signs and dividing, $x=3$.

Taking the negative sign, $x^2-18x=-x^2+54$.

Transposing and dividing, $x^2-9x=27$.

Completing the square, $x^2-9x+\frac{81}{4}=27+\frac{81}{4}=\frac{189}{4}$.

Evolving, $x-\frac{9}{2}=\frac{3\sqrt{21}}{2}$.

Transposing, $x=\frac{9\pm3\sqrt{21}}{2}$.

Hence $x=3$, or $\frac{9\pm3\sqrt{21}}{2}$. The sum of these is 12.

5. Given $x^3+x^2=-4$ to find the value of x.

Ans. -2, or $\frac{1\pm\sqrt{-7}}{2}$.

6. Given $7x^2=x^3+36$ to find the values of x.

Ans. $x=6$, or 3, or -2.

7. Given $x^3-4x^2=-9$ to find the values of x.

Ans. $x=3$, or $\frac{1\pm\sqrt{13}}{2}$.

8. Given $2x^3=99-5x^2$ to find the values of x.

Ans. $x=3$, or $\frac{-11\pm\sqrt{-143}}{4}$.

9. Given $4x^3+10x^2=125$ to find the values of x.

Ans. $x=2\frac{1}{2}$, or $\frac{-5\pm5\sqrt{-1}}{2}$.

10. Given $x^3=8x^2+363$ to find the values of x.

Ans. $x=11$, or $\frac{-3\pm\sqrt{-123}}{2}$.

11. Given $37x^2=7x^3+144$ to find the values of x.

Ans. $x=4$, or 3, or $-1\frac{5}{7}$.

CUBIC EQUATIONS CONTAINING ONLY THE THIRD AND FIRST POWERS.

Art. 344. Any numerical equation containing only the third and first powers of the unknown quantity, and having one rational root, may be reduced by multiplying both sides of the equation by the unknown quantity, and adding the second power to each side, with such a coefficient as, after adding a number readily determined, will make them perfect squares. The only difficulty lies in finding this coefficient, which must be ascertained by trial; though, by adopting the following rule, it can readily be found, unless the equation is so complicated, or the numbers so large, as to render the operation tedious.

In this and also the preceding case, the rule might perhaps be so framed as to obtain the roots without reducing the coefficient of the cube to unity, the two methods bearing somewhat the same relation to each other as the two in quadratic equations. But we have preferred to use fractions occasionally, rather than render the rule more complicated.

Rule. *Divide both sides of the equation by the coefficient of the unknown cube,* if there be any expressed. Place the two powers of the unknown quantity on one side, and the known quantity on the other, and multiply both sides by the unknown quantity with such a sign as shall render the fourth power positive.*

Separate the coefficient of the first power of the unknown quantity in the equation, thus produced, into two factors, and add the second power, with a coefficient equal to the square of one of these factors, usually the smaller, to each side. If it make the

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* Until the factors are found, it is sometimes better to give the known quantity and the first power a common denominator, even though the former might be reduced to a whole number.

coefficient of the square, on the same side as the fourth power, equal to the other factor, add the square of half this coefficient to each side, and extract the square root of both members, completing the operation by former rules.

But, if the above coefficient be not equal to the other factor, separate the same number into two other factors, or perhaps exchange the same, and proceed in the same way till the right ones are found.

NOTE 1. — The sum of the three values of the unknown quantity should always be 0, as there is no second power in the original equation ; hence, if two are known, the third will be equal to their sum, with the sign changed ; and there must always be one positive and one negative value, the other being sometimes positive and sometimes negative.

2. We obtain the three values by the same method as in the preceding case, prefixing the sign ± to the right-hand member of the equation in evolving. Taking the positive sign, we obtain either one or two values, and the negative sign gives the remaining values or value. When one of the values is known, the others might also be found by bringing all the terms of the original equation to the same side, and dividing by the difference between the unknown quantity and its known value.

3. When two of the values are rational, the third will of course be rational ; and there may be three different methods of separating into factors, each of which will answer the purpose, thus giving three different solutions of the same equation.

EXAMPLES.

1. Given $x^3-3x=2$ to find the values of x.

Conditions, $x^3-3x=2$.

Multiplying by x, $x^4-3x^2=2x$.

Separating the coefficient of $2x$ into factors, 2×1.

Adding $(1)^2x^2$ to each side, $x^4-2x^2=x^2+2x$.

Add $(\frac{2}{2})^2$, $x^4-2x^2+1=x^2+2x+1$

Evolving, $x^2-1=\pm(x+1)$.

Taking the positive sign, and transposing, $x^2-x=2$.

By quadratics, $x=2$, or -1. *Ans.*

The sum of these is 1 ; hence the other value is -1, and the equation has two equal roots, -1, and -1.

2. Given $10x=x^3+3$ to find the values of x.

Conditions, $10x=x^3+3$.

Transposing, $-x^3+10x=3$.

Multiplying by $-x$, $x^4-10x^2=-3x$.

Separating into factors, $3=3\times1$.

Adding $(3)^2x^2$ to each side, $x^4-x^2=9x^2-3x$.

Since coef. of x^2 = the other factor,
add $(\frac{1}{2})^2$, $x^4-x^2+\frac{1}{4}=9x^2-3x+\frac{1}{4}$.

Evolving, $x^2-\frac{1}{2}=\pm(3x-\frac{1}{2})$.

Taking the positive sign, and cancelling, $x^2=3x$.

Dividing, $x=3$.

Taking the negative sign, $x^2-\frac{1}{2}=-3x+\frac{1}{2}$.

Transposing, $x^2+3x=1$.

By quadratics, $x=\dfrac{-3\pm\sqrt{13}}{2}$.

Hence, $x=3$, or $\dfrac{-3\pm\sqrt{13}}{2}$. The sum of these is 0.

3. Given $4x^3+3x=182$ to find the values of x.

Conditions, $4x^3+3x=182$.

Dividing by coefficient of x^3, $x^3+\dfrac{3x}{4}=\dfrac{182}{4}$.

Multiplying by x, $x^4+\dfrac{3x^2}{4}=\dfrac{182x}{4}$.

Separating $\dfrac{182}{4}$ into factors, $\dfrac{182}{4}=\dfrac{14}{4}\times13$.

Adding $\left(\dfrac{14}{4}\right)^2x^2$ to each side, $x^4+13x^2=\dfrac{49x^2}{4}+\dfrac{91x}{2}$.

Since coefficient of x^2 = the other factor,
add $\left(\dfrac{13}{2}\right)^2$, $x^4+13x^2+\left(\dfrac{13}{2}\right)^2=\dfrac{49x^2}{4}+\dfrac{91x}{2}+\left(\dfrac{13}{2}\right)^2$.

Evolving, $x^2+\dfrac{13}{2}=\pm\left(\dfrac{7x}{2}+\dfrac{13}{2}\right)$.

Taking the positive sign and cancelling, $x^2=\dfrac{7x}{2}$.

Dividing, $x=\frac{7}{2}$.

Taking the negative sign, $x^2+\frac{13}{2}=-\frac{7x}{2}-\frac{13}{2}$.

Transposing, $x^2+\frac{7x}{2}=-13$.

Completing the square, $x^2+\frac{7x}{2}+\frac{49}{16}=-13+\frac{49}{16}=-\frac{159}{16}$.

Evolving, $x+\frac{7}{4}=\pm\frac{\sqrt{-159}}{4}$.

Transposing, $x=\frac{-7\pm\sqrt{-159}}{4}$.

The sum of these values is 0.

4. Given $x^3-7x=6$ to find the values of x.

Ans. $x=3$, or -1, or -2.

5. Given $x^3=37x+84$ to find the values of x.

Ans. $x=7$, or -3, or -4.

6. Given $2x^3+7x=474$ to find the values of x.

Ans. $x=6$, or $\frac{-6\pm\sqrt{-122}}{2}$.

7. Given $9x^3=169x+280$ to find the values of x.

Ans. $x=5$, or $-\frac{7}{3}$, or $-\frac{8}{3}$.

8. Given $x^3-3x=322$ to find the values of x.

Ans. $x=7$, or $\frac{-7\pm3\sqrt{-15}}{2}$.

PROBLEMS.

1. There is a cubical block of marble; and if 50 be added to the number of square feet in half its surface, it will be equal to the number of cubic feet in its contents. What are the solid contents of the block?

Let x = the side of the cube.

Then, x^3 = the contents of the block.

And x^2 = the superficial contents of one side of the block.

Then, $3x^2$ = the superficial contents of one-half the surface of the block.

Therefore, $x^3=3x^2+50$.

Multiplying both sides by 8, $8x^3=24x^2+400$.

Adding x^4 and square of $4x$, $x^4+8x^3+16x^2=x^4+40x^2+400$.

Evolving, $x^2+4x=x^2+20$.

Cancelling, $4x=20$.

Dividing, $x=5$.

Therefore the contents, $x^3=125$ cubic.

2. A gentleman having asked a lady her age, she replied, that if 29 times the square of her age were subtracted from twice its cube, the remainder would be 225. What was the lady's age?

Let $x=$ lady's age.

Then, $2x^3-29x^2=225$.

Transposing, $2x^3=29x^2+225$.

Adding x^4 and the square of x, $x^4+2x^3+x^2=x^4+30x^2+225$.

Evolving, $x^2+x=x^2+15$.

Cancelling, $x=15$ years.

3. A boy, being asked what he gave for his books, replied, that if 51 times the square of the number of dollars he gave for them were subtracted from 6 times the cube of the number, the remainder would be 900. What was the price of the books?

Ans. $10.

4. A man, being asked how many dollars he had in his pockets, replied, that if three times the cube of the number he had in his pockets were added to five times the square of the number which he had, he should have 272. Required the number he had in his pockets. *Ans.* $4.

5. A boat has been sailing two hours, with a light breeze, against a strong current; nineteen times the number of miles it has sailed is equal to the cube of that distance, added to thirty miles. How far has it sailed? *Ans.* It has gained either 3 miles or 2 miles, or it has lost 5 miles.

MISCELLANEOUS QUESTIONS.

1. Multiply $7\sqrt{x+3}\sqrt{x^3}-\sqrt{x^n}$ by $0\sqrt{y^3}$. *Ans.*

2. Multiply a^2+b^2 by $a^{-2}-b^{-2}$. *Ans.* $a^{-2}b^2-a^2b^{-2}$.

3. Multiply a^m+b^n by $a^{-2m}+b^{-n}$.
Ans. $a^{-m}+a^{-2m}b^n+a^mb^{-n}+1$.

4. Multiply $\sqrt{ax}$ by $-\sqrt{ax}$. *Ans.* $-ax$.

5. Divide $-a$ by $-3a$. *Ans.* $\frac{1}{3}$.

6. Divide a^{n-m} by a^n. *Ans.* a^{-m}.

7. Divide a^5+x^5 by $a+x$. *Ans.* $a^4-a^3x+a^2x^2-ax^3+x^4$.

8. Multiply y^m+x^m by $y-x$.
Ans. $y^{m+1}+x^my-xy^m-x^{m+1}$.

9. Multiply $\frac{1}{x^n}-\frac{na}{x^{n+1}}+\frac{n^2a^2}{2x^{n+2}}$ by $x^n+nax^{n-1}+\frac{n^2a^2x^{n-2}}{2}$.
Ans. $1+\frac{n^4a^4}{4x^4}$.

10. Divide $1-x^8$ by $1-x$.
Ans. $1+x+x^2+x^3+x^4+x^5+x^6+x^7$.

11. Multiply $3\sqrt[3]{x-a^2}$ by $4\sqrt{x^2-a}$.
Ans. $12\sqrt[6]{(x^8-3ax^6-2a^2x^7+3a^2x^4+6a^3x^5+a^4x^6-a^3x^2-6a^4x^3}$
$-3a^5x^4+2a^5x+3a^6x^2-a^7.)$

12. Given $\frac{41-35x}{105}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{2x-2\frac{1}{5}}{6}$ to find the value of x. *Ans.* $x=4$.

13. Given $\sqrt{x+9}=1+\sqrt{x}$ to find the value of x.
Ans. $x=16$.

14. Given $\frac{3-2x}{1-2x}-\frac{5-2x}{7-2x}=1-\frac{4x^2-2}{7-16x+4x^2}$ to find the value of x. *Ans.* $x=-\frac{7}{8}$.

15. Given $(\sqrt{x+28})(\sqrt{x+6})=(\sqrt{x+36})(\sqrt{x+4})$ to find the value of x. *Ans.* $x=4$.

16. Given $(x-1)\sqrt{2x-x^2}=\frac{1}{2}$ to find x.

Ans. $x=\frac{\sqrt{2}\pm1}{\sqrt{2}}$.

17. Given $x-2\sqrt{x+2}=1+\sqrt[4]{x^3-3x+2}$ to find x.

Ans. $x=9\pm4\sqrt{7}$, or $\frac{3\pm\sqrt{13}}{2}$.

18. Given $\sqrt[m]{a+x}=\sqrt[2m]{x^2-5ax+b^2}$ to find the value of x.

Ans. $x=\frac{b^2-a^2}{7a}$.

19. Given $b^2=a^2+bx$ to find the value of x.

Ans. $x=\frac{b^2-a^2}{b}$.

20. Given $\frac{5}{6}(x-a)-\frac{1}{5}(2x-3b)=10a+11b$ to find the value of x.

Ans. $x=25a+24b$.

21. Given $\frac{3ac}{a+b}+\frac{ax}{(a+b)^2}+\frac{(2a+b)bx}{a(a+b)^2}=\frac{3cx}{b}+\frac{x}{a}$ to find the value of x.

Ans. $x=\frac{ab}{a+b}$.

22. Given $\frac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}=b^{\frac{1}{2}}$ to find the value of x.

Ans. $x=\frac{2ab^{\frac{1}{2}}}{1+b}$.

23. Given $\frac{(a+x^{\frac{1}{2}})^{\frac{1}{2}}}{x^{\frac{1}{4}}}+\frac{(a-x^{\frac{1}{2}})^{\frac{1}{2}}}{x^{\frac{1}{4}}}=x^{\frac{1}{4}}$ to find the value of x.

Ans. $x=4(a-1)$.

24. Given $\left(\frac{a^2}{x}+b\right)^{\frac{1}{2}}-\left(\frac{a^2}{x}-b\right)^{\frac{1}{2}}=c^{\frac{1}{2}}$ to find the value of x.

Ans. $x=\frac{4a^2c}{4b^2+c^2}$.

25. A gentleman travelled 252 miles. The first day he rode 4 miles, the last 128, and each day's journey was double the preceding one. How many days was he performing the journey?

Ans. 6 days.

26. A gentleman dying left his sons an estate of $13,187.50. He bequeathed to his youngest son $1000, to the oldest $5062.50,

and ordered that each son's portion should exceed the next younger by the ratio of $1\frac{1}{2}$. How many sons had he?

Ans. 5 sons.

27. The first term in a geometrical progression is 3, the last term $\frac{1}{9}$, and the sum of the series $4\frac{4}{9}$. What is the number of terms? *Ans.* 4.

28. The first term in a geometrical series is $\frac{1}{5}$, the ratio 7, and the last term $3361\frac{2}{5}$. What is the number of terms? *Ans.* 6.

29. What are the three arithmetical means between $\frac{1}{3}$ and $\frac{1}{2}$?

Ans. $\frac{3}{8}$, $\frac{5}{12}$, $\frac{11}{24}$.

30. Required the sum of 200 terms of the series 1, 3, 5, 7, 9, &c. *Ans.* 40,000.

31. The first term of an arithmetical series is -7, the tenth term is 12. What is the sum of the series? *Ans.* 25.

32. If a man travel 20 miles the first day, and 15 miles the second, and so continue to travel 5 miles less each day, how far will he have advanced on his journey the 8th day?

Ans. 20 miles.

33. The first term of an arithmetical series is 5, the number of terms 20; what must the common difference be, that the sum of the series shall be $123\frac{1}{2}$? *Ans.* $\frac{47}{380}$.

34. If a man travel 20 miles the first day, 19 the second day, $18\frac{1}{20}$ the third day, and so on in a geometrical progression, in how many days will he have travelled 400 miles? *Ans.* $\frac{1}{0}$.

35. A merchant, having mixed a certain number of gallons of wine and water, found that if he had mixed 6 gallons more of each, there would have been 7 gallons of wine to every 6 gallons of water; but, if he had mixed 6 gallons less of each, there would have been 6 gallons of wine to every 5 gallons of water. How much of each did he mix?

Ans. 78 gallons of wine with 66 of water.

36. A person bought 2 cubical stacks of hay for £41; each of them cost as many shillings per solid yard as there were linear yards in a side of the other, and the greater occupied

9 square yards of ground more than the less. What was the price of each? *Ans.* £25 and £16.

37. A certain man owes \$1000. What sum shall he pay daily, so as to cancel the debt, principal and interest, at the end of the year, reckoning simple interest at 6 per cent.?
Ans. \$2.81974.

38. A and B travelled on the same road, and at the same rate. from Portland to Boston. When A was at 50 miles' distance from Boston he overtook a drove of geese, which were proceeding at the rate of 3 miles in 2 hours; and, two hours afterwards, met a stage-wagon, which was moving at the rate of 9 miles in 4 hours. B overtook the same drove of geese when he was 45 miles distance from Boston, and met the stage-wagon exactly 40 minutes before he arrived within 31 miles of Boston Where was B when A arrived at Boston?
Ans. 25 miles from Boston.

39. A gentleman has two sons, John and Nathan. John is 10 years old, and Nathan is 15. He wishes to divide \$1000 between his sons, in such a manner that each, by depositing his share in a savings' bank which pays 5 per cent. compound interest, shall have the same amount in the bank when he is 21 years old. What sum shall each deposit?
Ans. John, \$439.30; Nathan, \$560.70.

40. My garden is 100 feet square, and I wish to raise its surface 2 feet with the soil taken from a ditch with which I intend to surround it. This ditch is to be 5 feet deep, and outside the garden; what should be its width? *Ans.* 9.1+ feet.

41. A engaged to reap a field for \$10, which he would do in 10 days; but after he had labored 2 days he engaged B, by whose aid he supposed he could finish the field in 3 days. But, B proving to be a very inefficient workman, A was obliged to hire C the last two days, who proved to be a superior laborer; the field was completed in 5 days. Now, if he had not hired C, and A and B had completed the work themselves, B would have received \$1.08$\frac{8}{19}$ in addition to his services for his 3 days'

labor. How long would it have required B and C, each, to reap the field?

Ans. B could have reaped it in $11\frac{1}{9}$ days, C in $8\frac{16}{23}$ days.

42. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he would have been 6 hours longer in performing the same journey. How many miles did he go per hour? *Ans.* 7 miles.

43. The difference between the hypothenuse and base of a right-angled triangle is 6 feet, and the difference between the hypothenuse and perpendicular is 3 feet. What are the sides of the triangle? *Ans.* 15, 12, and 9 feet.

44. In a parcel which contains 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins, and each copper coin is worth as many pence as there are silver coins, and the whole is worth 18 shillings. How many are there of each? *Ans.* 6 of one, and 18 of the other.

45. The income of a certain estate is to be sold for a term of 7 years. A offers to pay $300 *down*, and $300 at the end of each year; B offers $800 *down*, and $250 at the end of each year; C offers $1300 *down*, and $200 at the end of each year; D will pay $2500 "*cash down.*" Which has made the best offer, if interest is to be reckoned at 6 per cent. compound interest?

Ans. { Value of A's offer, $1974.71.4.
B's offer, $2195.59.5; C's offer, $2416.47.6.
D's offer, $2500. Hence D's offer is the best.

46. A gentleman being asked the age of his two sons, replied, that if the sum of their ages were multiplied by the age of the elder, the product would be 144; but if the difference of their ages were multiplied by that of the younger, the product would be 14. What was the age of each? *Ans.* 9 and 7.

47. The sum of two numbers is 20, and the sum of their cubes is 2060. What are the numbers? *Ans.* 9 and 11.

48. If the product of two numbers be added to the square of the larger, the sum will be 112; but, if the square of the less

be taken from their product, the remainder will be 12. Required the numbers. *Ans.* 8 and 6.

49. What number is that which, being added to twice its square root, equals 24? *Ans.* 16.

50. If a man owe $2000, what sum shall he pay daily, so as to cancel the debt, principal and interest, at the end of the year, reckoning the interest at 6 per cent.? *Ans.* $5.6394.

51. I have 84½ square feet of plank, that is 3 inches thick. How large a cubical box can be made from it?

Ans. Each side measures 48 inches.

52. From 62$\frac{25}{72}$ feet of plank, that is 2½ inches thick, I wish to make a box whose length shall be four times its width, and whose height and width shall be equal. What are its dimensions? *Ans.* Length 8 feet, width and height 2 feet.

53. There was a cask containing 20 gallons of wine; a certain quantity of this was drawn off into another cask of equal size, and this last filled with water, and afterwards the first cask was filled with the mixture. It now appears that, if 6⅔ gallons of the mixture be drawn off from the first into the second cask, there will be equal quantities of wine in each. What was the quantity of wine drawn off at first? *Ans.* 10 gallons.

54. After A had travelled for 2¾ hours, at the rate of 4 miles an hour, B set out to overtake him; and, in order thereto, went four miles and a half the first hour, four and three-quarters the second, five the third, and so on, gaining a quarter of a mile every hour. In how many hours would he overtake A?

Ans. 8 hours.

55. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth is 60. Required the numbers. *Ans.* 5, 10, 20, 40.

56. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

Ans. 14, 10, 6, 2.

57. A tailor bought a piece of cloth for £147, from which he cut off 12 yards for his own use; he sold the remainder for £120 5*s*., gaining 5 shillings per yard. How many yards were there, and what did it cost him per yard?

Ans. 49 yards, at £3 per yard.

58. In a mixture of rye and wheat, the difference between the quantities of each is to the quantity of wheat as 100 is to the number of bushels of rye, and the same difference is to the quantity of rye as 4 to the number of bushels of wheat. How many bushels are there of each?

Ans. 25 bushels of rye, and 5 of wheat.

59. It is required to find two numbers, such that the product of the greater into the square root of the less shall be equal to 48, and the product of the less into the square root of the greater may be 36. *Ans.* 16 and 9.

60. If the difference of two numbers be multiplied by the greater, and the product divided by the less, the result will be 48; but, if the difference be multiplied by the less, and the product divided by the greater, the result will be 3. What are the numbers? *Ans.* 16 and 4.

61. Find two numbers, such that the square of the greater, multiplied by the less, shall be equal to 448; and the square of the less, multiplied by the greater, shall be 392.

Ans. 8 and 7.

62. If two numbers be each multiplied by 27, the first product is a square, and the second the square root of that square; but, if each be multiplied by 3, the first product is a cube, and the second the cube root of that cube. What are the numbers?

Ans. 243 and 3.

63. A farmer has two cubical stacks of hay; the side of one is 3 yards longer than the side of the other, and the difference of their contents is 117 solid yards. Required the side of each.

Ans. 5 and 2 yards.

64. A gentleman started from Boston for New York; he travelled 20 miles the first day, 18 miles the second day, and 16

miles the third day, so continuing to travel two miles less each day than the former. How far was the gentleman from Boston at the end of the twentieth day? *Ans.* .

65. A certain farm is a parallelogram, and a diagonal line from one corner to the opposite is 60 rods, and the longer side is to the shorter as 4 to 3. Required the contents of the farm. *Ans.* 10 Acres, 3 Roods, 8 Poles.

66. A gentleman asking a lady her age, she replied, If you add the square root of it to half of it, and subtract 12, there will remain nothing. Required her age. *Ans.* 16.

67. What number is that to which if 1, 7, and 19 be severally added, the first sum shall have the same ratio to the second that the second has to the third? *Ans.* 5.

68. The sum of two numbers is 12, and they have the same ratio to each other that their difference has to 40. What are the numbers? *Ans.* 2 and 10.

69. There are two numbers whose product is 54, and the greater is to the less as their sum is to 10. What are those numbers? *Ans.* 9 and 6.

70. Divide 20 into two such parts that the square of the greater shall be to the square of the less as 9 to 4. What are those parts? *Ans.* 12 and 8.

71. Let 24 be divided into two such parts that the quotient of the greater divided by the less shall be to the quotient of the less divided by the greater as 9 to 1. *Ans.* 18 and 6.

72. Divide 14 into two such parts that their squares shall be to each other as 9 to 16. *Ans.* 6 and 8.

73. Divide 12 into two such parts that the sum of their squares shall be to the difference of their squares as 5 to 3. *Ans.* 4 and 8.

74. There are two numbers, whose product is 12, and the sum of whose cubes is to the cube of their sum as 91 to 343. What are the numbers? *Ans.* 3 and 4.

75. The product of two numbers is 120; and, if the greater be increased by 8 and the less by 5, the product of the two numbers will be 300. What are the numbers? *Ans.* 10 and 12.

76. A, B, and C, make a joint stock; A puts in \$60 less than B, and \$68 more than C, and the sum of the shares of A and B is to the sum of the shares of B and C as 5 to 4. What did each put in?

Ans. A put in \$140, B \$200, and C \$72.

77. A and B engage in speculation, with different sums; A gains \$150, B loses \$50. Now A's stock is to B's as 3 to 2; but, had A lost \$50, and B gained \$100, then A's stock would have been to B's as 5 to 9. What was the stock of each?

Ans. A's, \$300; B's, \$350.

78. Find two numbers in the ratio of 5 to 7, to which two other numbers, in the ratio of 3 to 5, being respectively added, the sums shall be in the ratio of 9 to 13, and the difference of the sums shall be 16.

Ans. { First two numbers, 30 and 42.
Last two numbers, 6 and 10.

79. A merchant mixes wheat, which costs 10 shillings per bushel, with barley, which costs him 4 shillings per bushel, in such proportion as to gain 43¾ per cent., by selling the mixture at 11 shillings per bushel. Required the proportion.

Ans. He must mix 14 bushels of wheat with 9 of barley.

80. A and B can dig a cellar in a days, A and C can do the labor in b days, and B and C can do the same in c days. In what time would each perform the labor, and how long would it require A, B, and C, to complete the work?

Ans. A in $\frac{2abc}{ac+bc-ab}$ days, B in $\frac{2abc}{ab+bc-ac}$ days, C in $\frac{2abc}{ab+ac-bc}$ days, and A, B, C, in $\frac{2abc}{ab+ac+bc}$ days.

81. A and B made a joint stock of \$833, which, after a successful speculation, produced a clear gain of \$153. Of this B had \$45 more than A. What did each person contribute to the stock? *Ans.* B \$539, and A \$294.

82. A gentleman having asked a lady her age, she modestly replied, that if she were four years younger, and he were four years older, his age would be twice that of hers; but, if she were four years older, and he were four years younger, their ages would be the same. What was the age of each?

Ans. Gentleman's age, 28 years; lady's age, 20 years.

ALGEBRA APPLIED TO GEOMETRY.

83. Suppose a tree, 48 feet in height, to stand on a horizontal plane. At what height from the ground must it be cut off, so that the top of it may fall on a point 24 feet from the bottom of the tree, the end, where it was cut off, resting on the stump? *Ans.* 18 feet.

84. A certain man, owning a farm lying in a circle, gave it in his will to his wife, four sons, and four daughters, as follows: to his sons he gave four circles, as large as could be drawn within the circumference of the farm; to his daughters he gave the four spaces lying between the son's circles and the circumference of the farm, and to his wife he gave the part remaining in the centre, which contained just one acre. How much did the whole farm contain, how much did each son have, and how much did each daughter have?

Ans. { The farm contained 21 Acres, 1 Rood, 12 Poles.
Each son had 3 Acres, 2 Roods, 25½ Poles.
Each daughter had 1 Acre, 1 Rood, 27½ Poles.

85. A gentleman has a garden in the form of an equilateral triangle, the sides whereof are each 100 feet. At each corner of the garden stands a tower; the height of the first tower is 40 feet, that of the second 45 feet, and that of the third is 55 feet. At what distance from the bottom of each of these towers must a ladder be placed, that it may just reach the top of each tower; and what must be the length of the ladder, the ground of the garden being horizontal?

Ans. From the foot of the ladder to the base of the first tower, 63.273+ feet; second tower, 59.820+ feet; third tower, 50.779+ feet; length of the ladder, 74.856+ feet.

86. If c be the hypothenuse of a right-angled triangle, b the base, and a the perpendicular, it is required to find the segments made by a perpendicular drawn from the right angle to the hypothenuse.

Ans. $\frac{b^2+c^2-a^2}{2c}$, and $\frac{a^2+c^2-b^2}{2c}$.

87. From a point within an equilateral triangle, there are drawn three perpendiculars to the several sides; the length of the first is 20 feet, the second 30 feet, and the third 36 feet. Required the length of the sides of the triangle.

Ans. 49.652+ feet.

88. A sphere of gold, whose diameter is one inch, weighs 10 ounces, and each ounce is valued at $16. What is the value of 5 spheres of gold, whose several diameters are 1, 2, 3, 4, and 5 inches? *Ans.* $3600.

89. There is a loaf of bread, which is half a sphere, whose diameter measures 12 inches. How thick must the crust be baked, that the remainder shall be half the contents of the loaf?

Ans. .8038+ inch.

90. There are two towers of unequal heights, situated on a plane, near each other. A line extending from the base of the less to the top of the larger is 100 feet; and a line from the base of the larger to the top of the less is 80.27+ feet; a perpendicular let fall from the point where the lines cross each other, to the surface of the plane, is 32 feet. Required the height of the towers, and their distance from each other.

Ans. { Height of the larger tower, 80 feet.
Height of the less, 53⅓ feet.
Distance between the towers, 60 feet.

91. There is a conical glass, 6 inches deep; the diameter at the top is 5 inches, and it is ⅓ full of water. If a ball 4 inches in diameter be put into this glass, how much of its axis will be immersed in the water? *Ans.* .546 inch.

92. How many balls 1 inch in diameter can be put into a cubical box whose sides measure each one foot in the *clear?*

Ans. 2151 balls.

A

TABLE,

CONTAINING THE

LOGARITHMS OF NUMBERS

FROM 1 TO 10,000.

Numbers from 1 to 100 and their Logarithms, with their Indices.

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	0.000000	21	1.322219	41	1.612784	61	1.785330	81	1.908485
2	0.301030	22	1.342423	42	1.623249	62	1.792392	82	1.913814
3	0.477121	23	1.361728	43	1.633468	63	1.799341	83	1.919078
4	0.602060	24	1.380211	44	1.643453	64	1.806180	84	1.924279
5	0.698970	25	1.397940	45	1.653213	65	1.812913	85	1.929419
6	0.778151	26	1.414973	46	1.662758	66	1.819544	86	1.934498
7	0.845098	27	1.431364	47	1.672098	67	1.826075	87	1.939519
8	0.903090	28	1.447158	48	1.681241	68	1.832509	88	1.944483
9	0.954243	29	1.462398	49	1.690196	69	1.838849	89	1.949390
10	1.000000	30	1.477121	50	1.698970	70	1.845098	90	1.954243
11	1.041393	31	1.491362	51	1.707570	71	1.851258	91	1.959041
12	1.079181	32	1.505150	52	1.716003	72	1.857332	92	1.963788
13	1.113943	33	1.518514	53	1.724276	73	1.863323	93	1.968483
14	1.146128	34	1.531479	54	1.732394	74	1.869232	94	1.973128
15	1.176091	35	1.544068	55	1.740363	75	1.875061	95	1.977724
16	1.204120	36	1.556303	56	1.748188	76	1.880814	96	1.982271
17	1.230449	37	1.568202	57	1.755875	77	1.886491	97	1.986772
18	1.255273	38	1.579784	58	1.763428	78	1.892095	98	1.991226
19	1.278754	39	1.591065	59	1.770852	79	1.897627	99	1.995635
20	1.301030	40	1.602060	60	1.778151	80	1.903090	100	2.000000

NOTE.—In the following part of the Table the Indices are omitted, as they can be very easily supplied by the directions given in Section XXIX., p. 270, on Logarithms.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	000434	000868	001301	001734	002166	002598	003029	003461	003891	432
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
2	8600	9026	9451	9876	010300	010724	011147	011570	011993	012415	424
3	012837	013259	013680	014100	4521	4940	5360	5779	6197	6616	420
4	7033	7451	7868	8284	8700	9116	9532	9947	020361	020775	416
5	021189	021603	022016	022428	022841	023252	023664	024075	4486	4896	412
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
7	9384	9789	030195	030600	031004	031408	031812	032216	032619	033021	404
8	033424	033826	4227	4628	5029	5430	5830	6230	6629	7028	400
9	7426	7825	8223	8620	9017	9414	9811	040207	040602	040998	397
110	041393	041787	042182	042576	042969	043362	043755	044148	044540	044932	393
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
2	9218	9606	9993	050380	050766	051153	051538	051924	052309	052694	386
3	053078	053463	053846	4230	4613	4996	5378	5760	6142	6524	383
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	060320	379
5	060698	061075	061452	061829	062206	062582	062958	063333	063709	4083	376
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
7	8186	8557	8928	9298	9668	070038	070407	070776	071145	071514	370
8	071882	072250	072617	072985	073352	3718	4085	4451	4816	5182	366
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	079543	079904	080266	080626	080987	081347	081707	082067	082426	360
1	082785	083144	083503	3861	4219	4576	4934	5291	5647	6004	357
2	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
3	9905	090258	090611	090963	091315	091667	092018	092370	092721	093071	352
4	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
5	6910	7257	7604	7951	8298	8644	8990	9335	9681	100026	346
6	100371	100715	101059	101403	101747	102091	102434	102777	103119	3462	343
7	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
8	7210	7549	7688	8227	8565	8903	9241	9579	9916	110253	338
9	110590	110926	111263	111599	111934	112270	112605	112940	113275	3609	335
130	113943	114277	114611	114944	115278	115611	115943	116276	116608	116940	333
1	7271	7603	7934	8265	8595	8926	9256	9586	9915	120245	330
2	120574	120903	121231	121560	121888	122216	122544	122871	123198	3525	328
3	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
4	7105	7429	7753	8076	8399	8722	9045	9368	9690	130012	323
5	130334	130655	130977	131298	131619	131939	132260	132580	132900	3219	321
6	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
7	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
8	9879	140194	140508	140822	141136	141450	141763	142076	142389	142702	314
9	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	146438	146748	147058	147367	147676	147985	148294	148603	148911	309
1	9219	9527	9835	150142	150449	150756	151063	151370	151676	151982	307
2	152288	152594	152900	3205	3510	3815	4120	4424	4728	5032	305
3	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
4	8362	8664	8965	9266	9567	9868	160168	160469	160769	161068	301
5	161368	161667	161967	162266	162564	162863	3161	3460	3758	4055	299
6	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
7	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
8	170262	170555	170848	171141	171434	171726	172019	172311	172603	172895	293
9	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	176091	176381	176670	176959	177248	177536	177825	178113	178401	178689	289
1	8977	9264	9552	9839	180126	180413	180699	180986	181272	181558	287
2	181844	182129	182415	182700	2985	3270	3555	3839	4123	4407	285
3	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
4	7521	7803	8084	8366	8647	8928	9209	9490	9771	190051	281
5	190332	190612	190892	191171	191451	191730	192010	192289	192567	2846	279
6	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
8	8657	8932	9206	9481	9755	200029	200303	200577	200850	201124	274
9	201397	201670	201943	202216	202488	2761	3033	3305	3577	3848	272
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
160	204120	204391	204663	204934	205204	205475	205746	206016	206286	206556	271
1	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
2	9515	9783	210051	210319	210586	210853	211121	211388	211654	211921	267
3	212188	212454	2720	2986	3252	3518	3783	4049	4314	4579	266
4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
5	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
6	220108	220370	2[illegible]631	220892	221153	221414	221675	221936	222196	222456	261
7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
9	7887	8144	8400	8657	8913	9170	9426	9682	9938	230193	256
170	230449	230704	230960	231215	231470	231724	231979	232234	232488	232742	255
1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
2	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
3	8046	8297	8548	8799	9049	9299	9550	9800	240050	240300	250
4	240549	240799	241048	241297	241546	241795	242044	242293	2541	2790	249
5	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
7	7973	8219	8464	8709	8954	9198	9443	9687	9932	250176	245
8	250420	250664	250908	251151	251395	251638	251881	252125	252368	2610	243
9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	255273	255514	255755	255996	256237	256477	256718	256958	257198	257439	241
1	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
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6	9513	9746	9980	270213	270446	270679	270912	271144	271377	271609	233
7	271842	272074	272306	2538	2770	3001	3233	3464	3696	3927	232
8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
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1	281033	281261	281488	281715	281942	282169	2396	2622	2849	3075	227
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5	290035	290257	290480	290702	290925	291147	291369	291591	291813	292034	222
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4	9216	9401	9587	9772	9958	370143	370328	370513	370698	370883	185
5	371068	371253	371437	371622	371806	1991	2175	2360	2544	2728	184
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7	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
8	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
9	8398	8580	8761	8943	9124	9306	9487	9668	9849	380030	181
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6	390935	391112	391288	391464	391641	1817	1993	2169	2345	2521	176
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8	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
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2	401401	401573	1745	1917	2089	2261	2433	2605	2777	2949	172
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3	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
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9	460898	461048	461198	461348	461499	1649	1799	1948	2098	2248	150
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3	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
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5	9822	9969	470116	470263	470410	470557	470704	470851	470998	471145	147
6	471292	471438	1585	1732	1878	2025	2171	2318	2464	2610	146
7	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
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3	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
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5	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
6	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
7	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
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7	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
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5	550228	550351	550473	550595	550717	550840	550962	551084	551206	1328	122
6	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
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6	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
7	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
8	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
9	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
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2	570543	570660	570776	570893	571010	571126	1243	1359	1476	1592	117
3	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
4	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
5	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
6	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
7	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
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7	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
8	9883	9992	600101	600210	600319	600428	600537	600646	600755	600864	109
9	600973	601082	1191	1299	1408	1517	1625	1734	1843	1951	109
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2	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	108
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5	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
6	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	107
7	9594	9701	9808	9914	610021	610128	610234	610341	610447	610554	107
8	610660	610767	610873	610979	1086	1192	1298	1405	1511	1617	106
9	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
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5	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
6	9093	9198	9302	9406	9511	9615	9719	9824	9928	620032	104
7	620136	620240	620344	620448	620552	620656	620760	620864	620968	1072	104
8	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
9	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
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7	630428	630530	630631	630733	630835	630936	1038	1139	1241	1342	102
8	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
9	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
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3	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
4	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
5	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
6	9335	9432	9530	9627	9724	9821	9919	650016	650113	650210	97
7	650308	650405	650502	650599	650696	650793	650890	0987	1084	1181	97
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9	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
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4	7056	7152	7247	7843	7438	7534	7629	7725	7820	7916	96
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6	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
7	9916	660011	660106	660201	660296	660391	660486	660581	660676	660771	95
8	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718	95
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2	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
3	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	94
4	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	94
5	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
6	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
7	9317	9410	9503	9596	9689	9782	9875	9967	670060	670153	93
8	670246	670339	670431	670524	670617	670710	670802	670895	0988	1080	93
9	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	93
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1	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	92
2	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
3	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
4	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
5	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
6	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
7	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
8	9428	9519	9610	9700	9791	9882	9973	680063	680154	680245	91
9	680336	680426	680517	680607	680698	680789	680879	0970	1060	1151	91
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4	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
5	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
6	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
7	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
8	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
9	9309	9398	9486	9575	9664	9753	9841	9930	690019	690107	89
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1	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
2	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
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4	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
5	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	88
6	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
7	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
8	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
9	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
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3	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
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5	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
6	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
7	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
8	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
9	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
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8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
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5	720159	720242	720325	720407	720490	720573	720655	720738	720821	0903	83
6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
7	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
8	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
9	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
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1	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
2	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
3	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
5	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
7	9974	730055	730136	730217	730298	730378	730459	730540	730621	730702	81
8	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
9	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
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5	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
9	9572	9651	9731	9810	9889	9968	740047	740126	740205	740284	79
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1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
5	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
7	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
8	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
9	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
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3	750508	750586	750663	750740	0817	0894	0971	1048	1125	1202	77
4	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
5	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
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4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
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8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
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4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
5	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	770042	74
9	770115	770189	770263	770336	770410	770484	770557	770631	770705	0778	74
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1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
5	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
600	778151	778224	778296	778368	778441	778513	778585	778658	778730	778802	72
1	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	72
2	9596	9669	9741	9813	9885	9957	780029	780101	780173	780245	72
3	780317	780389	780461	780533	780605	780677	0749	0821	0893	0965	72
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
5	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
7	3189	3260	3332	3403	3475	3546	3618	8689	3761	3832	71
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
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1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
5	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
6	9581	9651	9722	9792	9863	9933	790004	790074	790144	790215	70
7	790285	790356	790426	790496	790567	790637	0707	0778	0848	0918	70
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
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1	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	70
2	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
5	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	799341	799409	799478	799547	799616	799685	799754	799823	799892	799961	69
1	800029	800098	800167	800236	800305	800373	800442	800511	800580	800648	69
2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
5	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
9	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
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1	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
4	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
5	9560	9627	9694	9762	9829	9896	9964	810031	810098	810165	67
6	810233	810300	810367	810434	810501	810569	810636	0703	0770	0837	67
7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	812913	812980	813047	813114	813181	813247	813314	813381	813448	813514	67
1	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
5	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
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9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
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1	820201	820267	820333	820399	820464	820530	820595	0661	0727	0792	66
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3	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
5	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
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2	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
5	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
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7	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
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9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
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2	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
3	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	64
4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
5	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
7	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
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1	9478	9541	9604	9667	9729	9792	9855	9918	9981	840043	63
2	840106	840169	840232	840294	840357	840420	840482	840545	840608	0671	63
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4	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
5	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
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2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
5	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
8	850033	850095	850156	850217	850279	850340	850401	850462	850524	850585	61
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
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4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
5	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
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7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
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1	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
2	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
5	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
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1	9818	9877	9935	9994	870053	870111	870170	870228	870287	870345	59
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3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
5	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
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9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
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8	9669	9726	9784	9841	9898	9956	880013	880070	880127	880185	57
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2	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
3	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
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5	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
6	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
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2	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
3	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
5	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
6	9862	9918	9974	890030	890086	890141	890197	890253	890309	890365	56
7	890421	890477	890533	0589	0645	0700	0756	0812	0868	0924	56
8	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
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3	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
5	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
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1	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
2	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
3	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
4	9821	9875	9930	9985	900039	900094	900149	900203	900258	900312	55
5	900367	900422	900476	900531	0586	0640	0695	0749	0804	0859	55
6	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
7	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
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4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
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6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
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5	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
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2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
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4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
5	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
6	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	919078	919130	919183	919235	919287	919340	919392	919444	919496	919549	52
1	9601	9653	9706	9758	9810	9862	9914	9967	920019	920071	52
2	920123	920176	920228	920280	920332	920384	920436	920489	0541	0593	52
3	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
5	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
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2	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
4	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
5	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
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8	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
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5	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
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